# A classification of multiple antenna channels

Joseph J. Boutros, Fatma Kharrat-Kammoun, and Hugues Randriambololona
Ecole Nationale Supérieure des Télécommunications, Paris 75013, France
Email: {boutros,fkharrat,randriam}@enst.fr

***Abstract*— We propose a new classification of multiple antenna channels. The classification is performed in the space of Hermitian forms defined by the channel representation. We introduce a geodesic metric between Hermitian forms and we build a finite number of classes. The centroids of the classification are found by a Generalized Lloyd algorithm. Numerical examples illustrate the geodesic orbits, capacity and error rate for 2 and 4 antennas.**

## I. Introduction

The material described in this paper can be applied to any problem that admits a lattice representation. The great interest of the research community and public in multiple antenna digital transmissions [1] made us focus the application of our Hermitian forms classification on that particular subject. The classification of multiple-input multiple-output (MIMO) frequency non-selective fading channels has a considerable importance in information and communication theory. Some potential applications would be adaptive modulation and adaptive channel coding in wireless local area networks and in 3G-4G mobile radio data networks. Other applications may also be related to transmitter and receiver algorithmic design. The classification algorithm given in this paper is valid for both types of frequency non-selective MIMO channels, i.e., channels with correlated and uncorrelated fading coefficients. The main procedure is illustrated in Fig. 1. Given the number of transmit and receive antennas, or equivalently, given the space dimension of Hermitian forms, given the number of classes to be distinguished, the classification algorithm runs in its training phase on a large number $M$ of Hermitian form training instances. Once a codebook of $K$ centroids is built, any instance $H$ can be quantized to the nearest class centroid. The classification needs a metric and an update rule as in the classical multidimensional Lloyd also known as k-Means clustering algorithm [3][6].
Let us briefly recall the mathematical model for a multiple antenna channel [1] and introduce some notations. We restrict our study to a square channel where the number $n_t$ of transmit antennas is equal to the number $n_r$ of receive antennas. Let $n = n_t = n_r$. The input-output model is given by

$$r = Hz + \nu$$

where the transmitted vector $z$ belongs to $\mathbb{Z}[i]^n$, $Z[i]$ being the ring of Gaussian integers, $\nu \in \mathbb{C}^n$ is an additive white Gaussian noise, $H = [h_{ij}]$ is an $n \times n$ matrix defining the MIMO channel coefficients, and $r$ is the received vector. We suppose that $H$ is perfectly known at the receiver side. No channel state information is needed in the transmitter because, in most imaginable applications, the classification will be performed at the receiver side. A widely used model assumes that $h_{ij}$ is complex Gaussian distributed with zero mean and unit variance. In the sequel, we assume that $det(H) = 1$ which corresponds to adding a multiplicative factor to the signal-to-noise ratio (a simple shift when expressed in dB). The probability distribution of $\det(H)$ can be further taken into account in potential applications of the proposed classification. A matrix $H$ in $SL_n(\mathbb{C})$ generates a complex lattice $\Lambda(H)$ with a normalized fundamental volume [2]. The lattice $\Lambda(H)$ is associated to the Hermitian form $z^\dagger H^\dagger H z$ where $A^\dagger$ denotes the transpose conjugate of $A$. The equivalence classes of MIMO channels are discussed in the next section. Mainly, two channels are equivalent in the quotient set $SU(n)\backslash SL_n(\mathbb{C})$ if there exist a unitary matrix $U$ such that $H_2 = UH_1$. Based on the homogeneous property of the quotient set and its embedded Riemannian structure, a geodesic distance is introduced in section III. The complex cubic lattice $\mathbb{Z}[i]^n$ associated to the identity matrix will play the role of a reference Hermitian form.

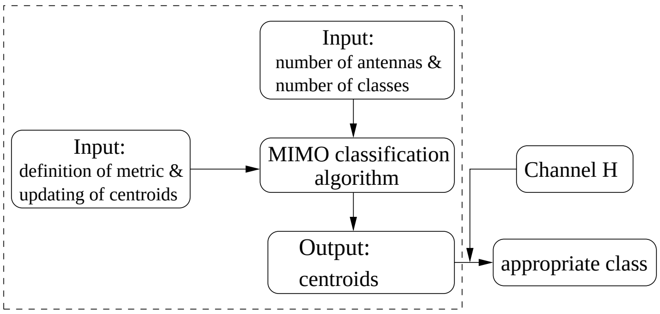


Fig. 1. The procedure of MIMO classification.

## II. Equivalence of MIMO channels

From a differential geometric point of view, the objects considered in this paper can be described as follows. Elements of $\mathbb{C}^n$ will be considered as column vectors, so that an unimodular basis of $\mathbb{C}^n$ corresponds to an element of the special linear group $SL_n(\mathbb{C})$, and the natural action of the special unitary group $SU(n)$ on $\mathbb{C}^n$ induces an action of $SU(n)$ on $SL_n(\mathbb{C})$, given by left multiplication. Thus the equivalence class of such a basis under unitary transformations corresponds to an element of the quotient set

$$SU(n)\backslash SL_n(\mathbb{C}).$$

Elements of this quotient set are cosets of the form $[H] = SU(n) \cdot H$ for $H \in SL_n(\mathbb{C})$. (Remark that if we had chosen line notation instead of column notation, then $SU(n)$ would have acted by right multiplication, and the quotient set to consider would have been $SL_n(\mathbb{C})/SU(n)$, whose elements are cosets of the form $H \cdot SU(n)$.)

We also mention another construction of related interest, although it will not be used in this paper. Two matrices in $SL_n(\mathbb{C})$ span the same lattice over $\mathbb{Z}[i]$ if and only if they differ by right multiplication by an element of $SL_n(\mathbb{Z}[i])$. Thus the set of equivalence classes of unimodular lattices over the ring of Gaussian integers up to unitary transformations in complex dimension $n$ can be identified with the double quotient space

$$SU(n)\backslash SL_n(\mathbb{C})/SL_n(\mathbb{Z}[i]),$$

whose elements are double cosets of the form $SU(n) \cdot H \cdot SL_n(\mathbb{Z}[i])$. The classification problem that will be treated in this paper for $SU(n)\backslash SL_n(\mathbb{C})$ could also be carried out for $SU(n)\backslash SL_n(\mathbb{C})/SL_n(\mathbb{Z}[i])$ with essentially the same techniques, although a few more tools are needed. This will be the subject of a forthcoming paper, so we do not enter into details here.

The quotient set $SU(n)\backslash SL_n(\mathbb{C})$ is an example of homogeneous space, that is, the quotient of a Lie group by a closed subgroup. As such, it carries a differential structure and an action of $SL_n(\mathbb{C})$, given by right multiplication. There is another way to describe this space that is quite useful in practice. Observe that two unimodular matrices $H_1$ and $H_2$ in $SL_n(\mathbb{C})$ have the same Gram matrix

$$G = H_1^\dagger H_1 = H_2^\dagger H_2$$

if and only if there is a unitary matrix $U \in SU(n)$ such that $H_2 = UH_1$, that is if and only if the classes of $H_1$ and $H_2$ in $SU(n)\backslash SL_n(\mathbb{C})$ coincide. Conversely, any positive definite Hermitian matrix $G$ of determinant 1 can be written in the form $G = H^\dagger H$ for $H \in SL_n(\mathbb{C})$ uniquely determined up to left multiplication by a unitary matrix (in fact one can mention two specific choices of $H$ that are of particular interest: the first one is to take $H$ upper triangular, given by Cholesky decomposition, while the second is to take $H$ Hermitian, by extracting the square root of $G$). These considerations allow to identify $SU(n)\backslash SL_n(\mathbb{C})$ with the space of positive definite Hermitian matrices of determinant 1, by identifying the coset $SU(n) \cdot H$ with the Gram matrix $G = H^\dagger H$. The differential structure on $SU(n)\backslash SL_n(\mathbb{C})$ can be retrieved as the natural differential structure on this space of Hermitian matrices seen as a real submanifold of $\mathbb{C}^{n\times n}$, and the right action of $SL_n(\mathbb{C})$ on this space can be described as follows: if $G$ is an Hermitian matrix and $P \in SL_n(\mathbb{C})$, then $P$ acts on $G$ by sending it to $P^\dagger GP$.

## III. Geodesic distance for Hermitian matrices

It appears in fact that $SU(n)\backslash SL_n(\mathbb{C})$ is a special type of homogeneous space, called a symmetric space, and thus carries a natural Riemannian structure. We will refer to [4] which is the classical reference on this topic (along with E. Cartan's original works). More precisely, following the classification given in [4], §IX.6.1, $SU(n)\backslash SL_n(\mathbb{C})$ is the Riemannian global symmetric space of type IV associated with the Lie algebra of type $\mathfrak{a}_{n-1}$. The aim of the following proposition is to describe the geodesics and the distance associated to this Riemannian structure.

*Proposition 1:* Let $G_1$ and $G_2$ be two positive definite Hermitian matrices of size $n$ with determinant 1. Let $P_1$ be any element of $SL_n(\mathbb{C})$ such that $G_1 = {P_1}^\dagger P_1$, and put $G = ({P_1}^\dagger)^{-1} G_2 {P_1}^{-1}$ and $L = \log(G)$. Then:

1) There is a unique geodesic segment $\gamma$ joining $G_1$ to $G_2$, given by the parameterization $\gamma(t) = {P_1}^\dagger \exp(tL) P_1$ for $t \in [0, 1]$.
2) Up to multiplication by a constant, the geodesic distance between $G_1$ and $G_2$ is

$$d_{\text{geod}}(G_1, G_2) = (\sum_{1\le i\le n} |\log \lambda_i|^2)^{1/2},$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $G$.

(Recall that if $A$ is an Hermitian matrix, and $F$ a real analytic function that is well defined on the eigenvalues of $A$, then $F(A)$ can be defined as follows: if $A = V^\dagger \mathrm{Diag}(\mu_1, ..., \mu_n)V$ is the diagonalization of $A$, with $V$ unitary and $\mu_1, ..., \mu_n$ real, then $F(A) = V^\dagger \mathrm{Diag}(F(\mu_1), ..., F(\mu_n))V$. In particular, with the notations of the proposition, if one writes $G = U^\dagger \mathrm{Diag}(\lambda_1, ..., \lambda_n)U$ with $U$ unitary, then $L = U^\dagger \mathrm{Diag}(\log(\lambda_1), ..., \log(\lambda_n))U$ and $\gamma(t) = {P_1}^\dagger U^\dagger \mathrm{Diag}(\lambda_1^t, ..., \lambda_n^t)UP_1$.)

*Proof:* Since the right action of $SL_n(\mathbb{C})$ must send geodesics to geodesics and preserve the distance, one can suppose that $G_1 = I_n$, $P_1 = I_n$, and $G_2 = G$.

Since $SU(n)$ is the set of fixed points of the analytic involution $\sigma : H \mapsto (H^\dagger)^{-1}$ of $SL_n(\mathbb{C})$, one can apply [4], Proposition IV.3.4, to retrieve the fact that $SU(n)\backslash SL_n(\mathbb{C})$ is a Riemannian globally symmetric space. The tangent space to $SL_n(\mathbb{C})$ at $I_n$ is easily seen to be the space of matrices with trace zero, and from [4], Proposition IV.3.3(iii), it follows that the tangent space $\mathcal{T}$ to $SU(n)\backslash SL_n(\mathbb{C})$ at $I_n$ can be identified with its subspace of anti-invariants under $d\sigma$. Since $d\sigma$ sends a matrix $M$ to $-M^\dagger$, one sees that this tangent space $\mathcal{T}$ is the space of Hermitian matrices with trace zero. Since $G = U^\dagger \mathrm{Diag}(\lambda_1, ..., \lambda_n)U$ is of determinant 1, one checks that $L = U^\dagger \mathrm{Diag}(\log(\lambda_1), ..., \log(\lambda_n))U$ indeed is an element of $\mathcal{T}$. Now it follows again from [4], Proposition IV.3.3(iii) that any geodesic segment starting at $I_n$ is of the form $\gamma_{L'} : t \mapsto \exp(tL')$ for some $L' \in \mathcal{T}$, and the condition $\gamma_{L'}(1) = G$ forces $L' = L$. This proves the first part of the proposition.

Now since $\gamma_L$ is a geodesic segment, its tangent vector has constant norm equal to $\|L\|$, where $\|.\|$ is the norm on $\mathcal{T}$ given by the Riemannian structure. This norm must be invariant under the action of $SU(n)$ on $\mathcal{T}$, and since this action is irreducible, $\|.\|$ is unique up to multiplication by a constant. One can check that the so-called Frobenius norm defined by

$$\|L\| = (\sum_{1\le i,j\le n} |L_{ij}|^2)^{1/2}$$

indeed is invariant. Using this invariance property, this can also written as:

$$\begin{aligned} d_{\text{geod}}(I_n, G) &= \text{length}(\gamma) \\ &= \|L\| = \|U^\dagger \text{Diag}(\log(\lambda_1), ..., \log(\lambda_n))U\| \\ &= \|\text{Diag}(\log(\lambda_1), ..., \log(\lambda_n))\| \\ &= (\sum_{1\le i\le n} |\log\lambda_i|^2)^{1/2}, \end{aligned}$$

which proves the second part of the proposition. ■

## IV. Centroids update in generalized Lloyd for Hermitian forms

The Generalized Lloyd algorithm iterates between two steps. The first step determines the borders of Voronoi regions. A Voronoi region is also called a class in our terminology. This Lloyd first step utilizes the geodesic metric given by proposition (1) in the previous section. The second step updates the centroid of each class.

In the second step of Lloyd's algorithm applied to the space $SU(n)\backslash SL_n(\mathbb{C})$, given some positive definite Hermitian matrices $G_1, ..., G_N$ with determinant 1, one needs to find the centroid $C$ of this class of cardinality $N$ that minimizes the sum of the squared distances $d_{\text{geod}}(C, G_1)^2 + ... + d_{\text{geod}}(C, G_N)^2$. To our knowledge, there is no exact way to perform this minimization, however, proposition (1) suggests a way to find at least a good approximation of this centroid, based on a gradient heuristic. Indeed, suppose we know that $G_1, ..., G_N$ already are not too far from an "old" centroid $C_0$. Using the invariance of the geodesic distance under the action of $SL_n(\mathbb{C})$, one can reduce to the case $C_0 = I_n$. Then if $L_i$ is the tangent vector at $I_n$ to the geodesic segment ending at $G_i$, so that $L_i = \log(G_i)$, the gradient at $I_n$ of the function $C \mapsto d_{\text{geod}}(C, G_1)^2 + ... + d_{\text{geod}}(C, G_N)^2$ is proportional to $L = L_1 + ... + L_N$. It is thus natural to take for $C$ the endpoint of the geodesic with tangent vector $\frac{1}{N}L$, that is

$$C = \exp(\frac{1}{N}(\log G_1 + ... + \log G_N)) \tag{1}$$

One can check easily that if $N = 1$, or if $N = 2$ and $C_0$ already lies on the unique geodesic passing through $G_1$ and $G_2$, then this approximate $C$ is the exact $C$ that minimizes the sum of the squared distances.

From a computational point of view, a drawback with these mathematical constructions is that they require to diagonalize some of the matrices involved, which can be heavily time-consuming. A computationally lighter alternative is then to consider the space of positive definite Hermitian matrices with determinant 1 as a subset of the affine space of all Hermitian matrices, and to use the distance and the averaging process coming from this affine structure. More precisely, this amounts to replacing the geodesic distance $d_{\text{geod}}$ with the Frobenius distance

$$d_{\text{lin}}(G_1, G_2) = (\sum_{1\le i,j\le n} |(G_1 - G_2)_{ij}|^2)^{1/2},$$

and to defining the linear average of $G_1, ..., G_N$ as $C' = \frac{1}{N}(G_1 + ... + G_N)$ and the new centroid its central projection $C = (\det C')^{-1/n} C'$.

The design of a quantizer for $SU(n)\backslash SL_n(\mathbb{C})$ via the different distances and these different centroids averaging processes should be tuned to the specific target application.

## V. Numerical results

We considered channels with uncorrelated coefficients before doing the determinant normalization. Our numerical results on MIMO classification are obtained via a Generalized Lloyd similar to the one-dimensional Lloyd [5].

1) Build an initial codebook with $K$ elements chosen randomly.
2) Assign each data sample to its nearest centroid (in the geodesic metric sense).
3) Update the centroid of class $i$ based on the $N_i$ data samples belonging to this class, for $i = 1, \ldots, K$. These are the new $K$ centroids.
4) Go back to step 2 during $n_L$ iterations.

Figures 2, 3, and 4 show the geodesic orbits of centroids after $n_L$ Lloyd iterations on $M = 10^6$ Hermitian form samples. The origin represents the identity matrix. A square symbol represents a centroid placed on a circle of radius equal to its geodesic distance to the identity. Angles surrounding centroids are proportional to the size of their classes. For some values of $K$ considered as small, the codebook is spherical in the geodesic representation and classes are equiprobable if Lloyd reaches a steady state at large $n_L$. Clearly, when $K$ is high, the codebook includes many orbits. Some singular orbits may correspond to rare Hermitian forms or to a non convergence state of Lloyd algorithm. Fig. 5 illustrates the capacity (log-det formula from [7]) versus SNR when $n = 2$. This figure reflects the 3 orbits found on Fig. 3. For higher dimensions, the same geodesic orbit may correspond to different channel capacities. Finally, Figures 6 and 7 show how the error rate behaves among $K = 256$ classes for $n = 4$ with a 16-QAM modulation and maximum-likelihood decoding [8].

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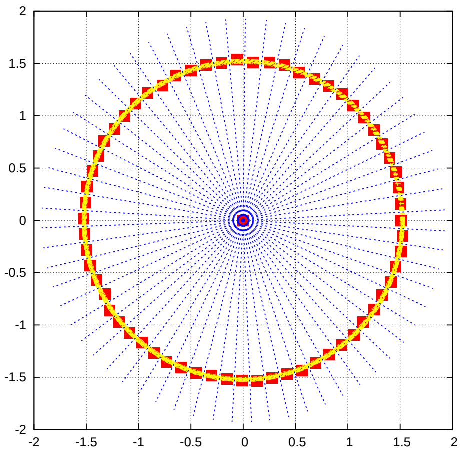

Fig. 2. Geodesic orbits for $n = 4$ antennas, $K = 64$ classes, $n_L = 32$ Lloyd iterations.

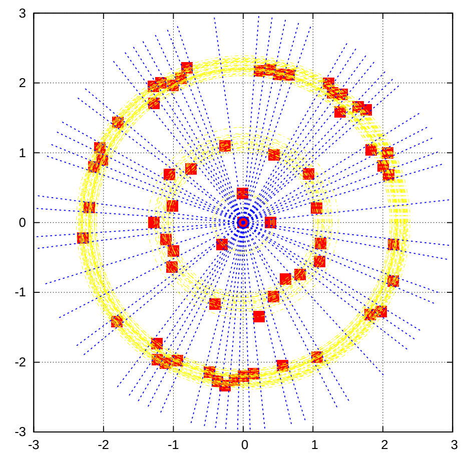

Fig. 3. Geodesic orbits for $n = 2$ antennas, $K = 64$ classes, $n_L = 50$ Lloyd iterations.

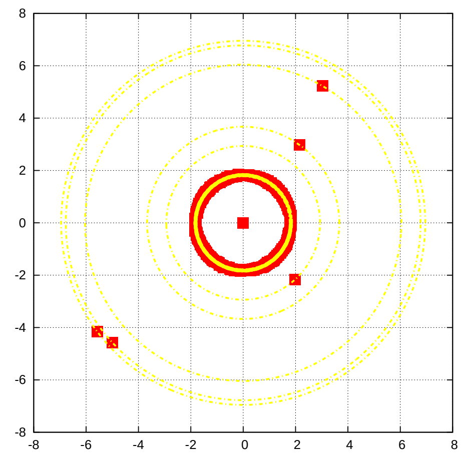

Fig. 4. Geodesic orbits for $n = 4$ antennas, $K = 256$ classes, $n_L = 32$ Lloyd iterations.

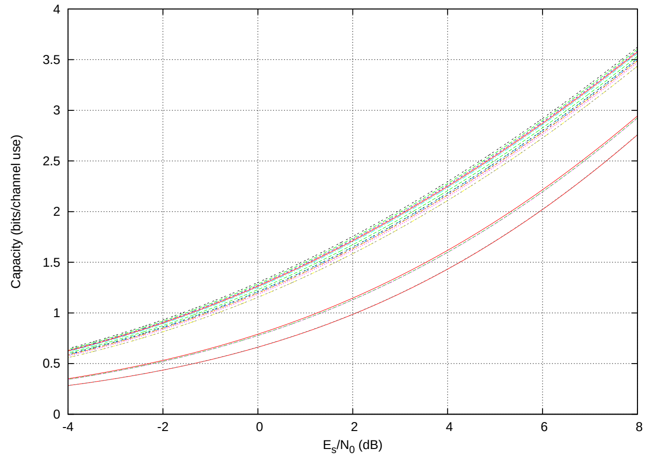


Fig. 5. Capacity versus signal-to-noise ratio for $n = 2$ antennas, $K = 64$ classes. The graph displays the capacity curve of 20 distinct classes.

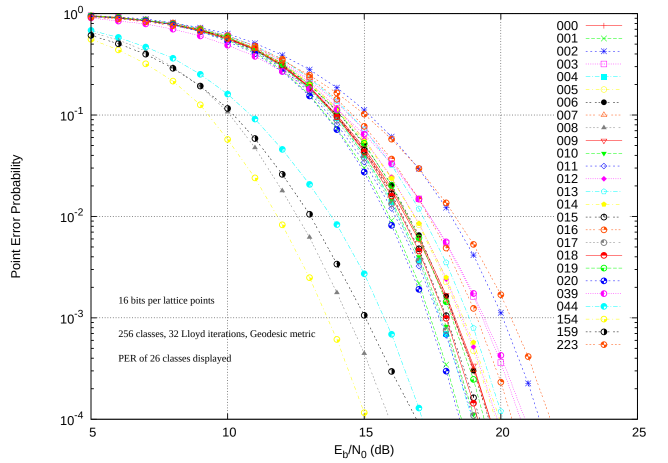


Fig. 6. Point error rate performance for $n = 4$ antennas, $K = 256$ classes. Components of $z$ are Gaussian integers taken from a 16-QAM constellation (uncoded).

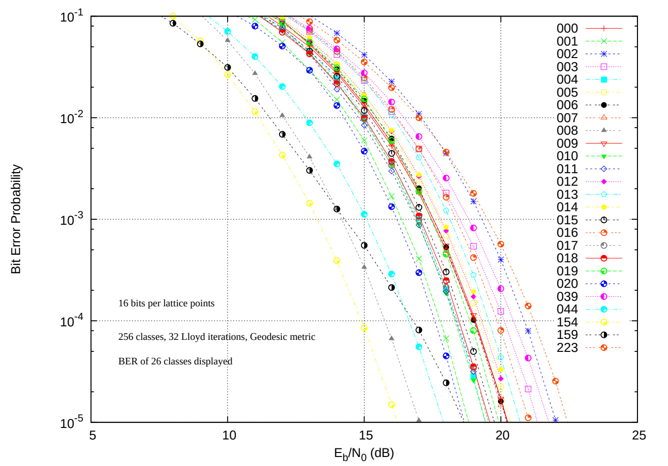


Fig. 7. Bit error rate (worst bit) performance for $n = 4$ antennas, $K = 256$ classes. Components of $z$ are Gaussian integers taken from a 16-QAM constellation (uncoded). The worst bit in the lattice constellation labeling is considered. The 16-QAM is Gray labeled.