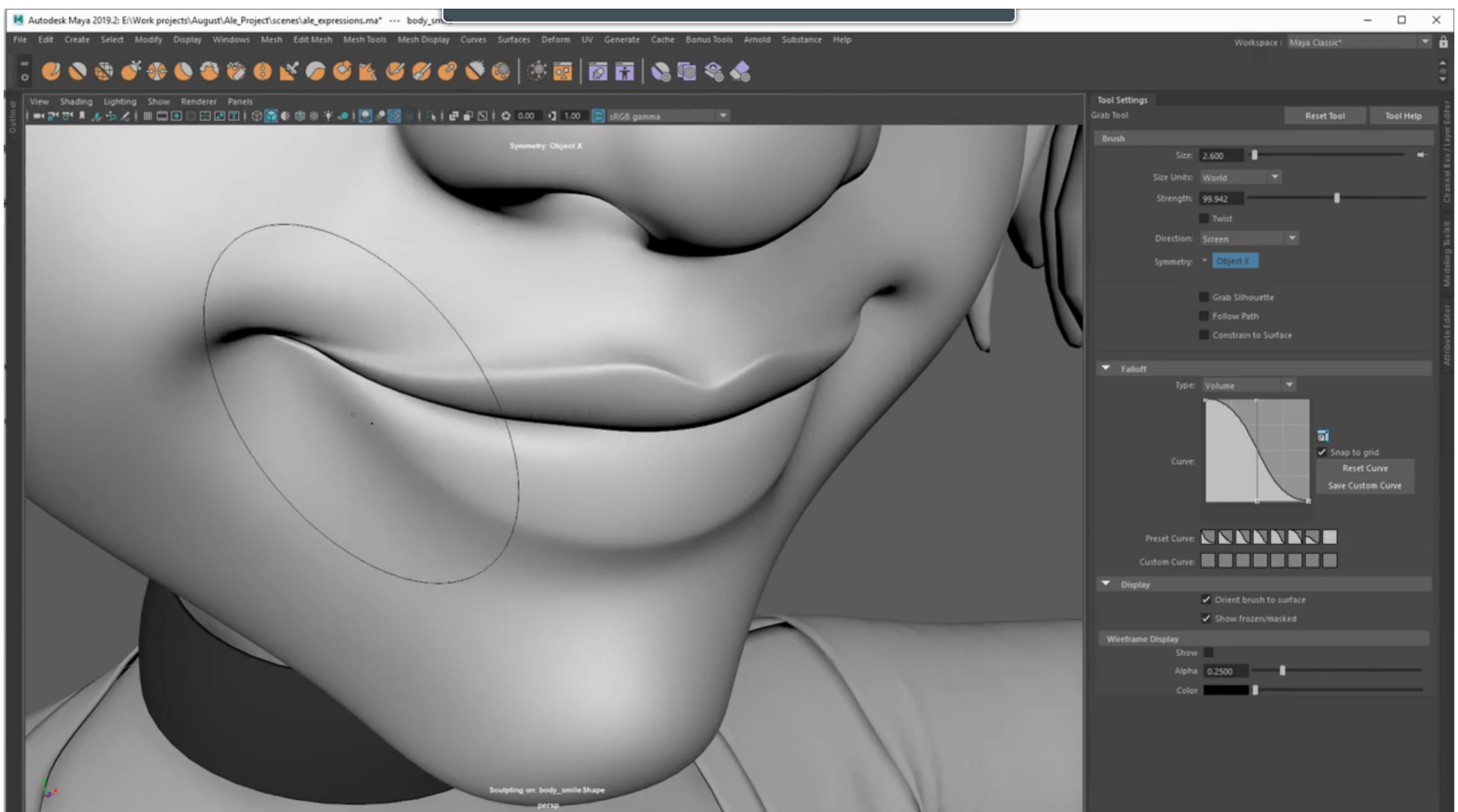
Modeling using brushes

Main idea

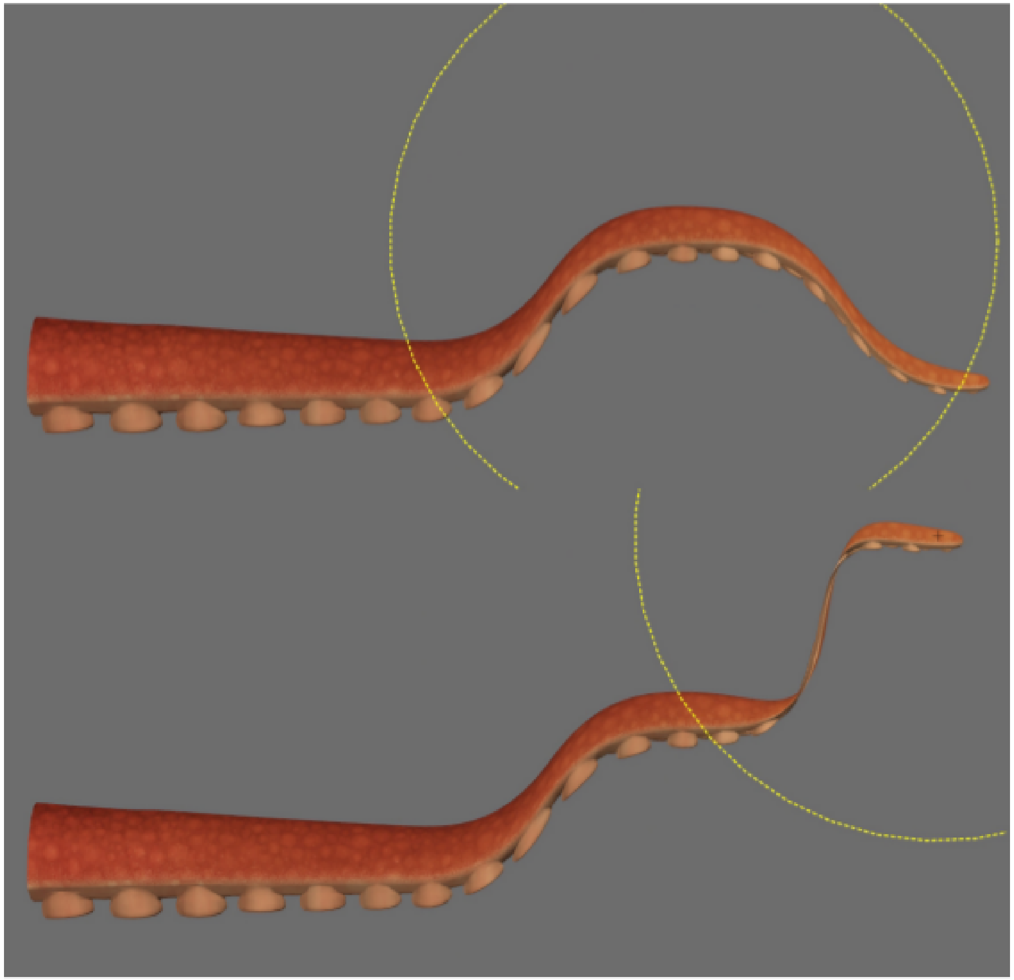
- Use a brush to define :
  - The type of transformation (set at the center of the brush)
  - The falloff (decay function describing the locality)



Standard types

- Types :
  - Grab / Translate
  - Rotate / Twist
  - Scale
  - Shear

- Modulate the translation using the falloff function
  - Creates artifacts
  - No control over volume
  - Simple to implement
  - Has been used forever

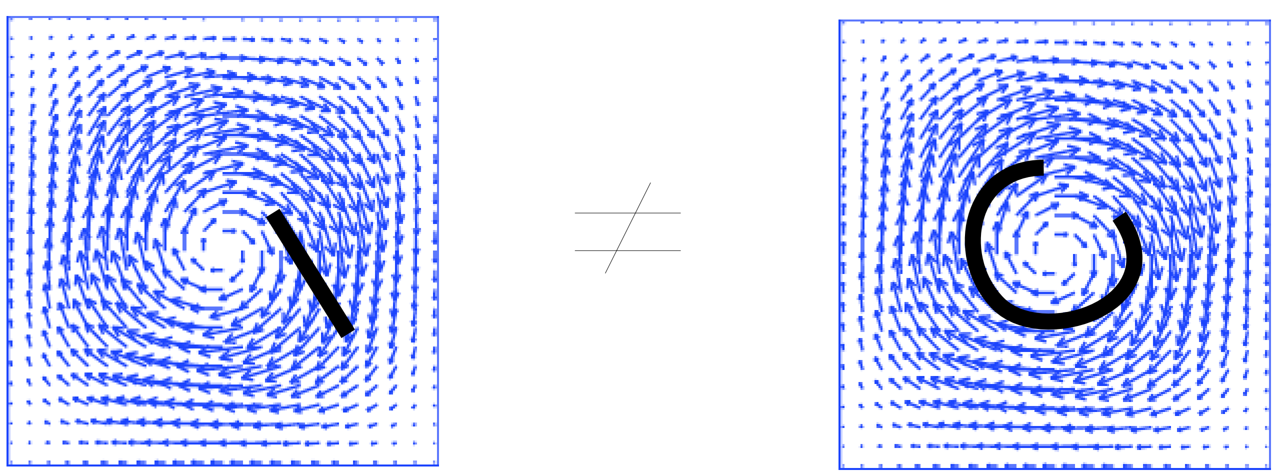


Classic grab brush

Regularized kelvinlets

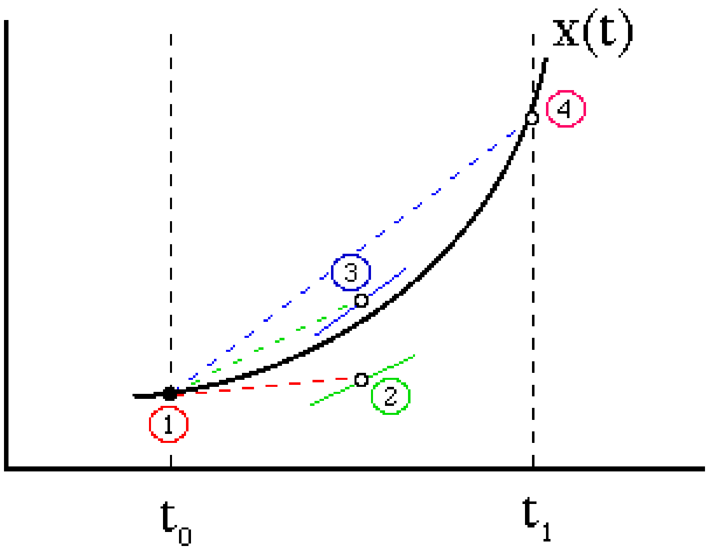
- Linked to incompressibility :
  - Use a vector field u(x) with 0 divergence (div(u)=0)
  - Advect it to obtain the deformation : dx/dt = u(x)
  - Deformations preserve volume (e.g., incompressible fluids)

- For standard brushes, a vector field u(x) is defined, and transformations are given by f(x) = x + u(x)
- Careful : not so simple anymore :



How to advect a (time-varying) vector field u(x ; t) for a time step h ?

- Explicit Euler : **x' = x + u(x ; t).h** very bad, poor convergence
- Implicit Euler : **x' = x + u(x' ; t+h).h** difficult to compute
- Runge Kutta (order 4) : $\mathbf{x' = x + h/6(k_1 + 2k_2 + 2k_3 + k_4)}$, with
  - $k_1 = u(x ; t)$
  - $k_2 = u(x + h/2 . k_1 ; t + h/2)$
  - $k_3 = u(x + h/2 . k_2 ; t + h/2)$
  - $k_4 = u(x + h . k_3 ; t + h)$



Used in most related works and in the Kelvinlets paper.
Approximation error : $O(h^5)$ at each step

Based on physics

Linear elasticity energy to minimize :

Minimize divergence (volume change)

$$E(\boldsymbol{u}) = \frac{\mu}{2}||\nabla \boldsymbol{u}||^2 + \frac{\mu}{2(1-2\nu)}||\nabla \cdot \boldsymbol{u}||^2 - \langle \boldsymbol{b}, \boldsymbol{u} \rangle$$

Minimize gradient (variation)

Maximize alignment with load **b**

Solution : (differentiate by u and set to 0)

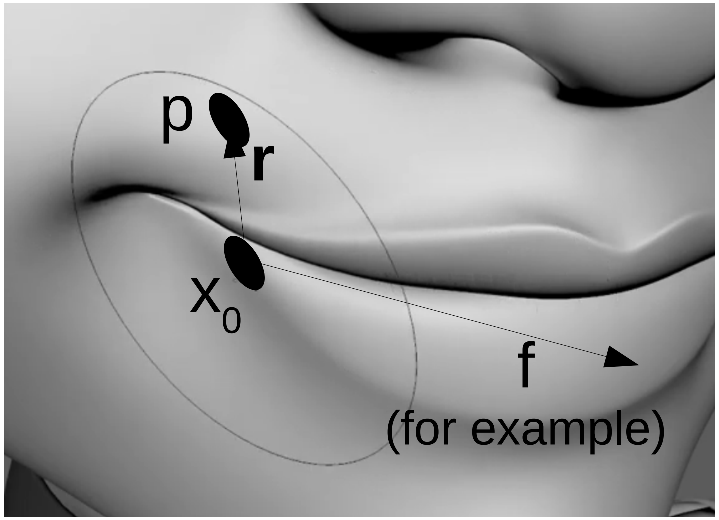
$$\mu \Delta \boldsymbol{u} + \frac{\mu}{(1-2\nu)} \nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{b} = 0$$

Solutions for grab brushes

$$\mu \Delta \boldsymbol{u} + \frac{\mu}{(1-2\nu)} \nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{b} = 0$$

Singular load (with singularity)
All the energy is concentrated in x_0

$$\boldsymbol{b}(\boldsymbol{x}) = f\, \underbrace{\delta(\boldsymbol{x} - \boldsymbol{x}_0)}_{\text{Dirac in } x_0}$$



$$\boldsymbol{u}(\boldsymbol{r}) = \left[\frac{(a-b)}{r} \boldsymbol{I} + \frac{b}{r^3} \boldsymbol{r}\boldsymbol{r}^t \right] \boldsymbol{f}$$

Singular for r=0
(r = |**r**| : distance from evaluation point p to the center of the brush x_0)

$$a = 1/(4\pi\mu) \qquad b = a/[4(1-\nu)].$$

Solutions for grab brushes

$$\mu \Delta \boldsymbol{u} + \frac{\mu}{(1-2\nu)} \nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{b} = 0$$

Singular load (w singularity)

$$\boldsymbol{b}(\boldsymbol{x}) = f\, \underline{\delta(\boldsymbol{x} - \boldsymbol{x}_0)}$$

Dirac in x_0

Replaced by

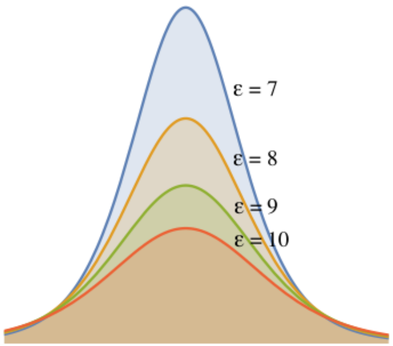
$$\boldsymbol{u}(\boldsymbol{r}) = \left[\frac{(a-b)}{r} \boldsymbol{I} + \frac{b}{r^3} \boldsymbol{r}\boldsymbol{r}^t \right] \boldsymbol{f}$$

Singular for r=0

Regularized

$$\rho_\varepsilon(\boldsymbol{r}) = \frac{15\varepsilon^4}{8\pi} \frac{1}{r_\varepsilon^7}$$

$$r_\varepsilon = \sqrt{r^2 + \varepsilon^2}$$



$$\boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \left[\frac{(a-b)}{r_\varepsilon} \boldsymbol{I} + \frac{b}{r_\varepsilon^3} \boldsymbol{r}\boldsymbol{r}^t + \frac{a}{2} \frac{\varepsilon^2}{r_\varepsilon^3} \boldsymbol{I} \right] \boldsymbol{f} \equiv \mathcal{K}_\varepsilon(\boldsymbol{r}) \boldsymbol{f}.$$

Not singular anymore ($r_\varepsilon > 0$)

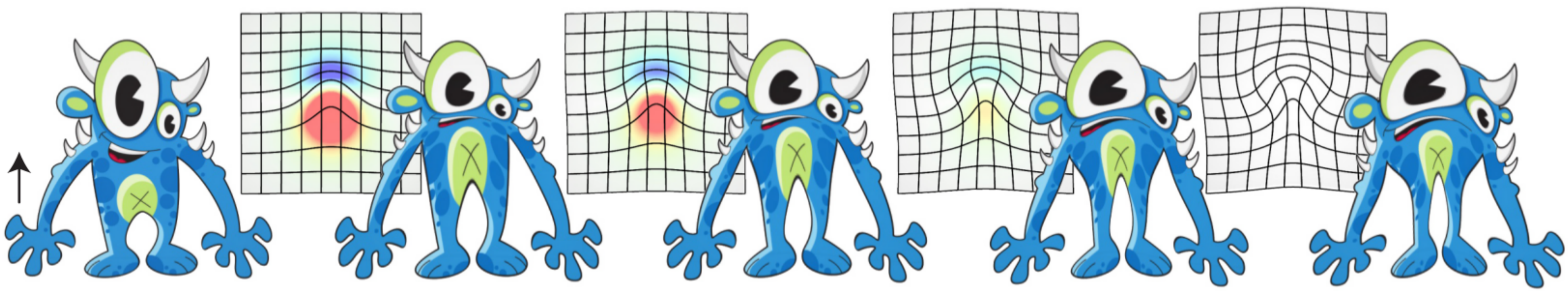
$$a = 1/(4\pi\mu) \qquad b = a/[4(1-\nu)].$$

Regularized grab brushes

$$r_\varepsilon = \sqrt{r^2 + \varepsilon^2}$$

$$\boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \left[\frac{(a-b)}{r_\varepsilon} \boldsymbol{I} + \frac{b}{r_\varepsilon^3} \boldsymbol{r}\boldsymbol{r}^t + \frac{a}{2} \frac{\varepsilon^2}{r_\varepsilon^3} \boldsymbol{I} \right] \boldsymbol{f} \equiv \mathcal{K}_\varepsilon(\boldsymbol{r}) \boldsymbol{f}.$$

component along **r**



Original $\quad \nu = -\infty \quad \nu = 0.0 \quad \nu = 0.4 \quad \nu = 0.5$

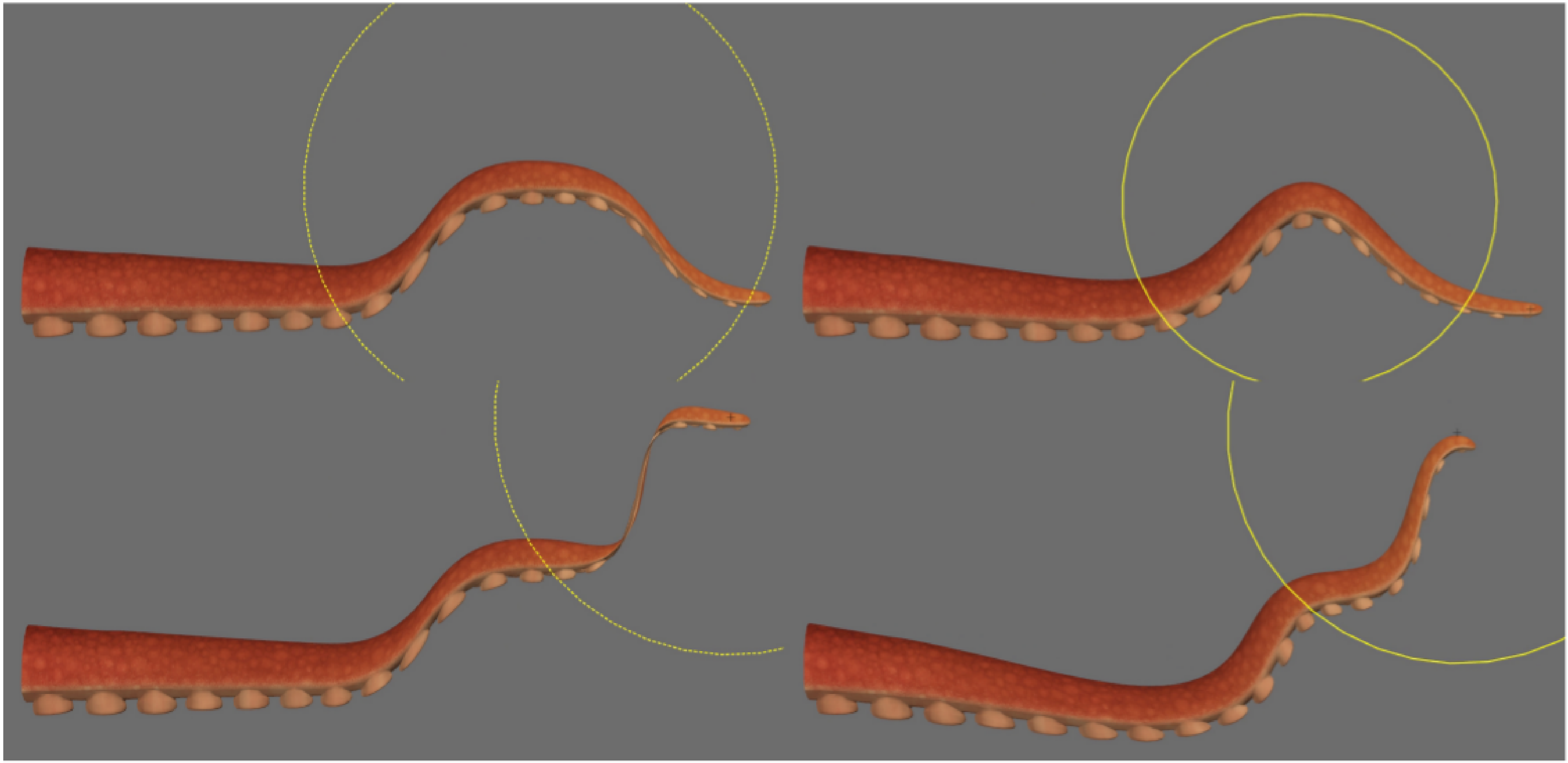
- **Note that the deformation is not restricted to the y axis !** (because of the component along **r**)
- Note that for a Poisson ratio = 0.5 (right image), the volume is exactly preserved everywhere.

Regularized grab brushes

$$r_\varepsilon = \sqrt{r^2 + \varepsilon^2}$$

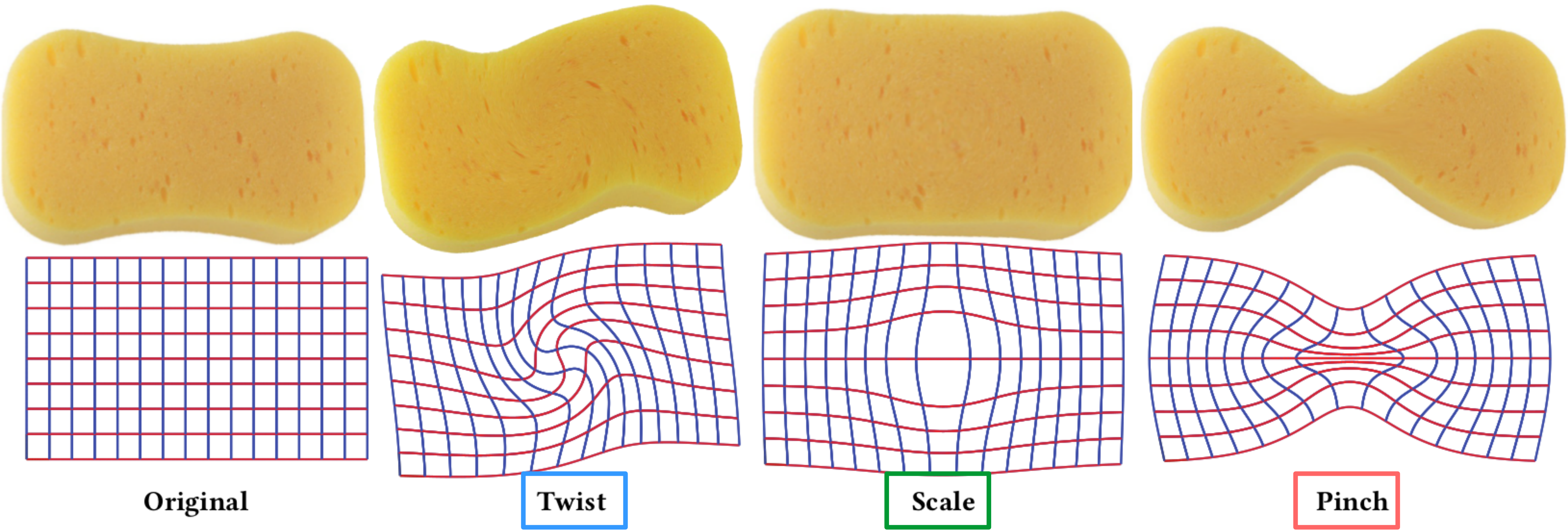
$$\boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \left[\frac{(a-b)}{r_\varepsilon} \boldsymbol{I} + \frac{b}{r_\varepsilon^3} \boldsymbol{r}\boldsymbol{r}^t + \frac{a}{2} \frac{\varepsilon^2}{r_\varepsilon^3} \boldsymbol{I} \right] \boldsymbol{f} \equiv \mathcal{K}_\varepsilon(\boldsymbol{r}) \boldsymbol{f}.$$

component along **r**



Classic grab brush **Regularized Kelvinlet**

Other regularized brushes



$$\boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \left[\frac{(a-b)}{r_\varepsilon}\boldsymbol{I} + \frac{b}{r_\varepsilon^3}\boldsymbol{r}\boldsymbol{r}^t + \frac{a}{2}\frac{\varepsilon^2}{r_\varepsilon^3}\boldsymbol{I} \right] \boldsymbol{f} \equiv \mathcal{K}_\varepsilon(\boldsymbol{r})\boldsymbol{f}.$$

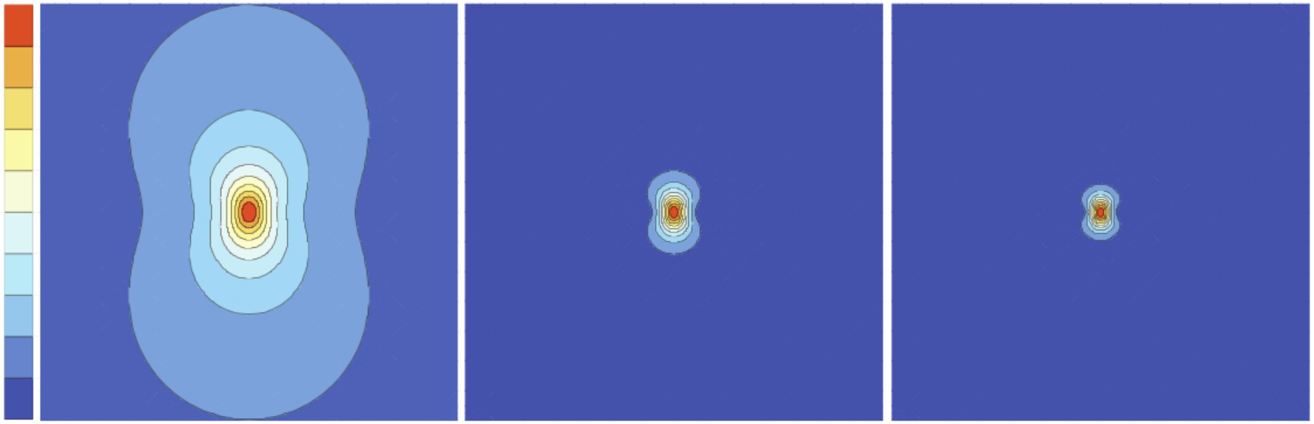
$$\boldsymbol{t}_\varepsilon(\boldsymbol{r}) = -a\left(\frac{1}{r_\varepsilon^3} + \frac{3\varepsilon^2}{2r_\varepsilon^5}\right)\boldsymbol{q} \times \boldsymbol{r}.$$

$$\boldsymbol{s}_\varepsilon(\boldsymbol{r}) = (2b-a)\left(\frac{1}{r_\varepsilon^3} + \frac{3\varepsilon^2}{2r_\varepsilon^5}\right)(s\ \boldsymbol{r}),$$

$$\boldsymbol{p}_\varepsilon(\boldsymbol{r}) = \frac{(2b-a)}{r_\varepsilon^3}\boldsymbol{F}\boldsymbol{r} - \frac{3}{2r_\varepsilon^5}\left[2\,b\,(\boldsymbol{r}^t\boldsymbol{F}\boldsymbol{r})\,\boldsymbol{I} + a\,\varepsilon^2\boldsymbol{F}\right]\boldsymbol{r}.$$

Multiscale brushed

- Kelvinlets are superposable (the PDE is linear)
- The standard Kelvinlet has a slow falloff (in 1/r)
- By superposing several kelvinlets with appropriate parameters, the falloff can be made faster (to better localize the influence)



Single-scale

Bi-scale

Tri-scale

$$\boldsymbol{u}_{\varepsilon_1,\varepsilon_2}(\boldsymbol{r}) = \left[\mathcal{K}_{\varepsilon_1}(\boldsymbol{r}) - \mathcal{K}_{\varepsilon_2}(\boldsymbol{r})\right] \boldsymbol{f}.$$

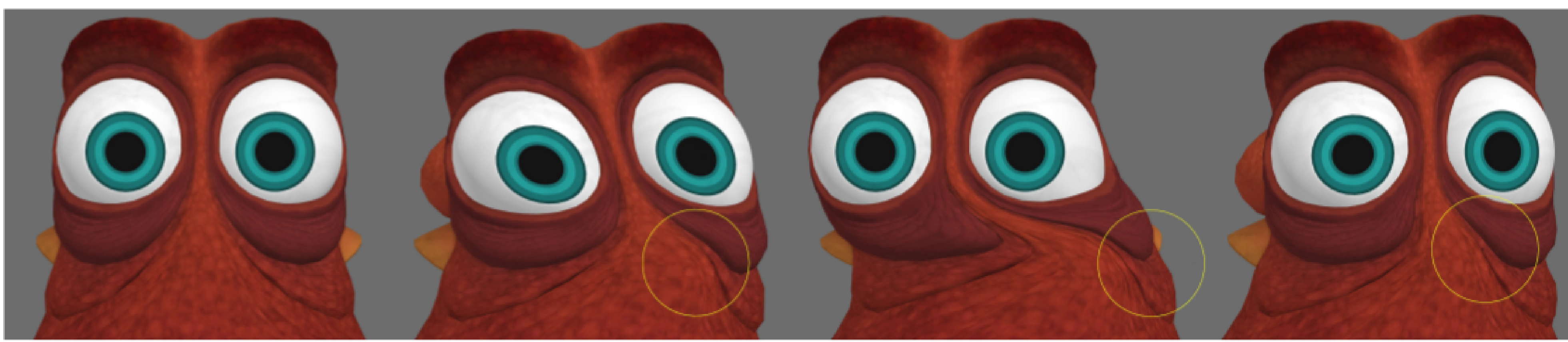
$$w_1 = 1,\ w_2 = -(\varepsilon_3^2-\varepsilon_1^2)/(\varepsilon_3^2-\varepsilon_2^2),\ w_3 = (\varepsilon_2^2-\varepsilon_1^2)/(\varepsilon_3^2-\varepsilon_2^2)$$

$$\boldsymbol{u}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(\boldsymbol{r}) = \left[w_1\mathcal{K}_{\varepsilon_1}(\boldsymbol{r}) + w_2\mathcal{K}_{\varepsilon_2}(\boldsymbol{r}) + w_3\mathcal{K}_{\varepsilon_3}(\boldsymbol{r})\right] \boldsymbol{f}$$

$$\varepsilon_{i+1} = 1.1\varepsilon_i$$

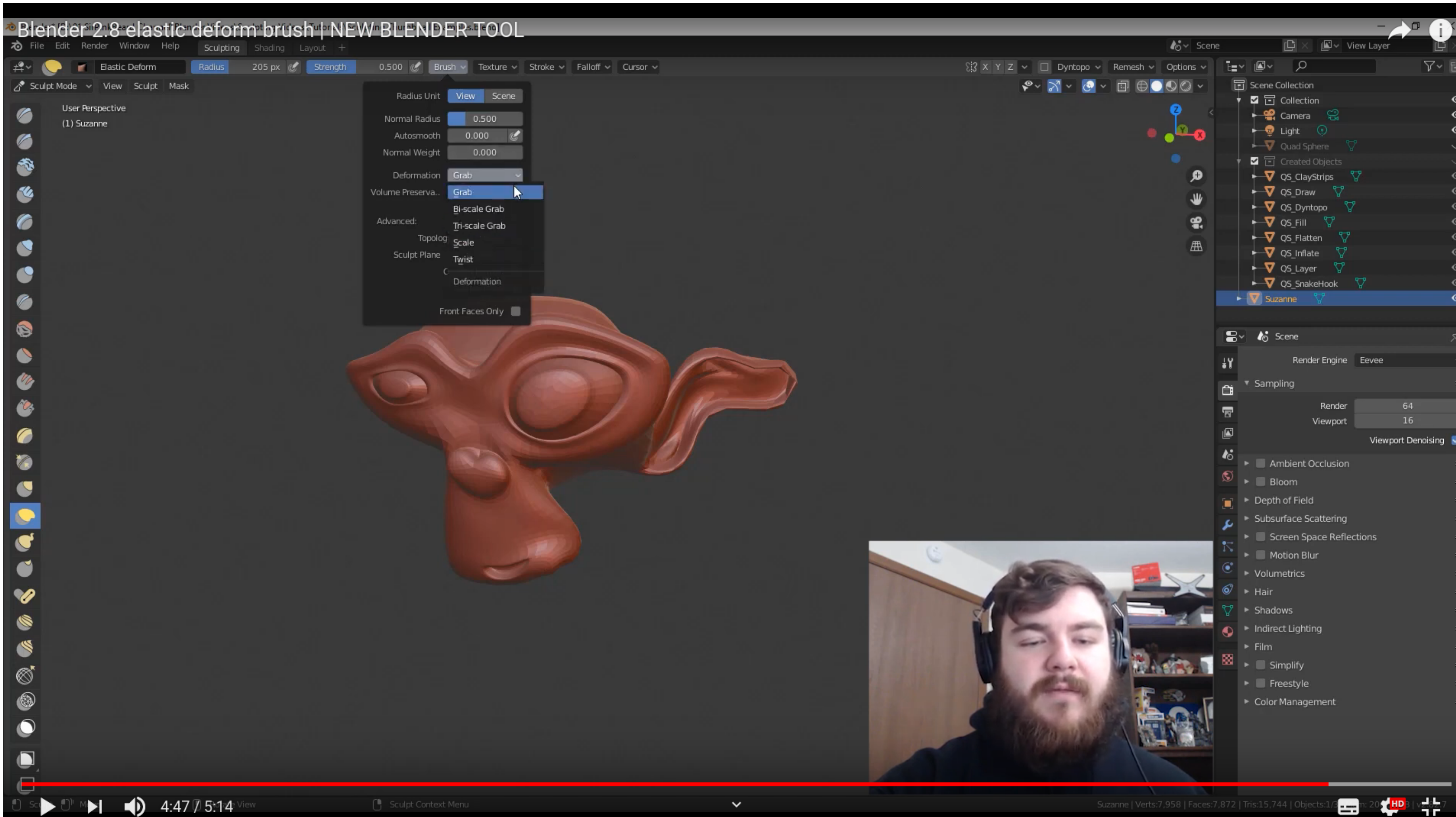
Constrained deformations

- Use additional Kelvinlets that you superpose
- Optimize the parameters to respect some constraints (requires setting up linear systems only)



Rest **Unconstrained** $\boldsymbol{u}(\boldsymbol{x}_i) = 0$ $\nabla \boldsymbol{u}(\boldsymbol{x}_i) = 0$

Demo in Blender 2.8



https://www.youtube.com/watch?v=TKzfy-NLVRs

Dynamic Kelvinlets

- Based on elastodynamics :

$$m\,\partial_{tt}\boldsymbol{u} = \mu\Delta\boldsymbol{u} + \frac{\mu}{(1-2\nu)}\nabla(\nabla\cdot\boldsymbol{u}) + \boldsymbol{b}$$

$$\mho_\gamma(r,t) = \frac{1}{16\pi\gamma r^3}\Big[\mathcal{W}(r,r+\gamma t) - \mathcal{W}(r,r-\gamma t)\Big],$$

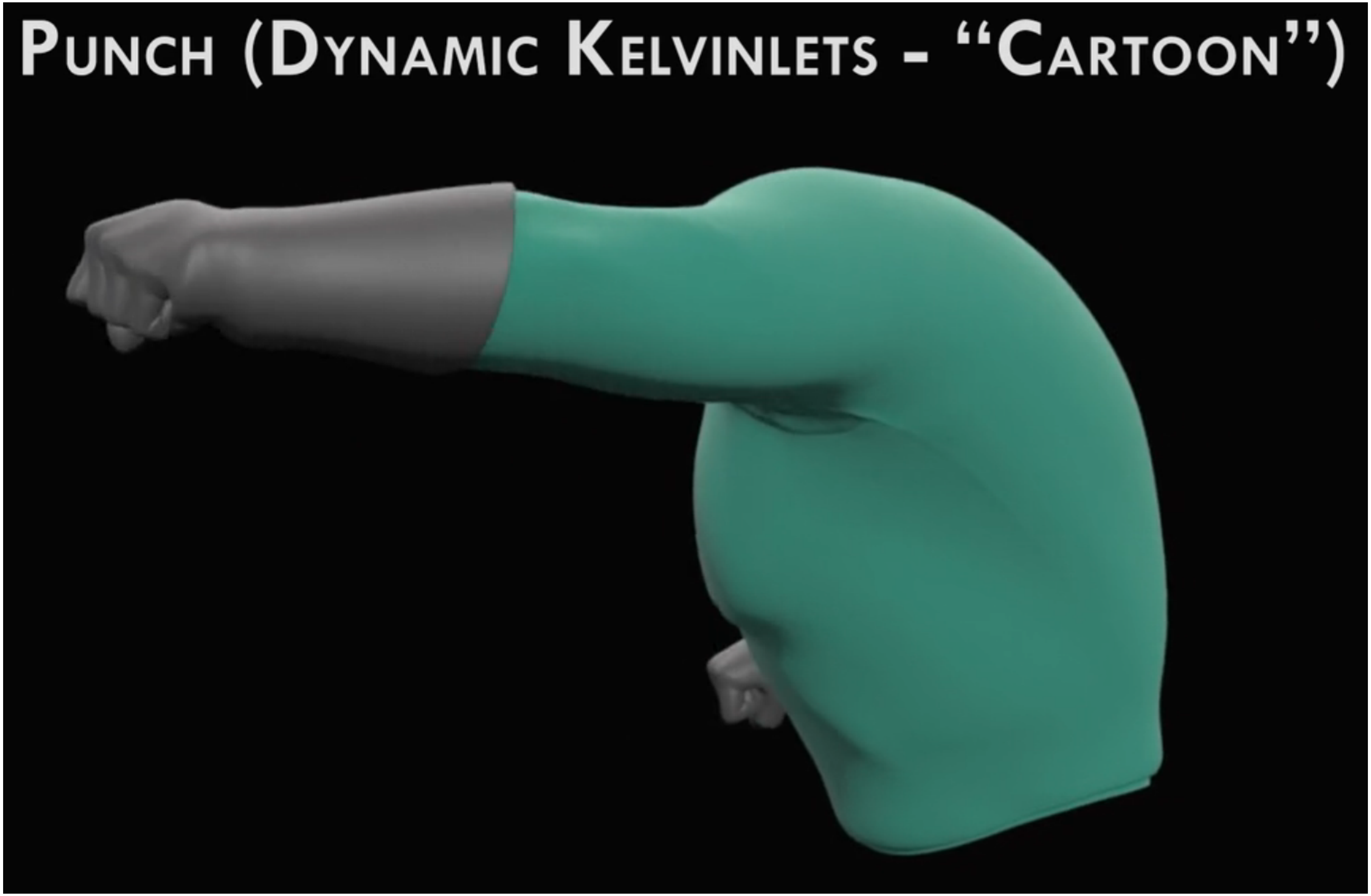
$$\mathcal{W}(r,s) = \frac{1}{s_\varepsilon}\left(2s^2+\varepsilon^2-3rs\right) + \frac{1}{s_\varepsilon^3}\,r\,s^3.$$

$$\boldsymbol{u}(\boldsymbol{r},t) = \Big[\mathcal{A}(r,t)\,\boldsymbol{I} + \mathcal{B}(r,t)\,\boldsymbol{r}\boldsymbol{r}^\top\Big]\boldsymbol{f} \equiv \mathcal{D}(\boldsymbol{r},t)\boldsymbol{f},$$

$$\mathcal{A}(r,t) = \mho_\alpha(r,t) + 2\mho_\beta(r,t) + r\,\partial_r\mho_\beta(r,t),$$

$$\mathcal{B}(r,t) = \Big(\partial_r\mho_\alpha(r,t) - \partial_r\mho_\beta(r,t)\Big)\,/r.$$

Dynamic Kelvinlets video



https://vimeo.com/269027205

See also the last ones if you are interested (earlier references)

- Refs :
  - *F. de Goes, D.L. James*. 2017. Regularized Kelvinlets: Sculpting Brushes based on Fundamental Solutions of Elasticity.
  - *F. de Goes, D.L. James*. 2018. Dynamic Kelvinlets: Secondary Motions based on Fundamental Solutions of Elastodynamics.
  - *W. von Funck, H. Theisel, and H. P. Seidel*. 2006. Vector field based shape deformations.
  - *W. von Funck, H. Theisel, and H. P. Seidel*. 2007. Elastic Secondary Deformations by Vector Field Integration.