Shannon's Formula $W \cdot \log(1 + SNR)$: A Historical Perspective

on the occasion of Shannon's Centenary

Oct. 26th, 2016

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Institut Henri Poincaré

Outline

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Claude Shannon (1916–2001)

100th birthday 2016

April 30, 1916 Claude Elwood Shannon was born in Petoskey, Michigan, USA

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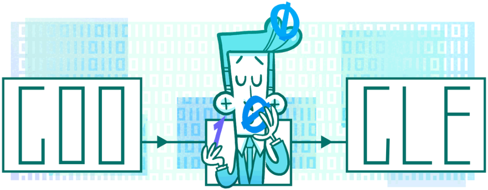

here Shannon is juggling with bits (1,0,0)
in his communication scheme

"father of the information age"

Do you Know Claude Shannon?

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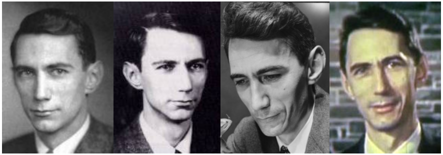

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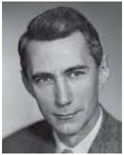

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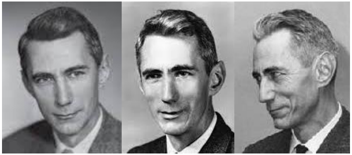

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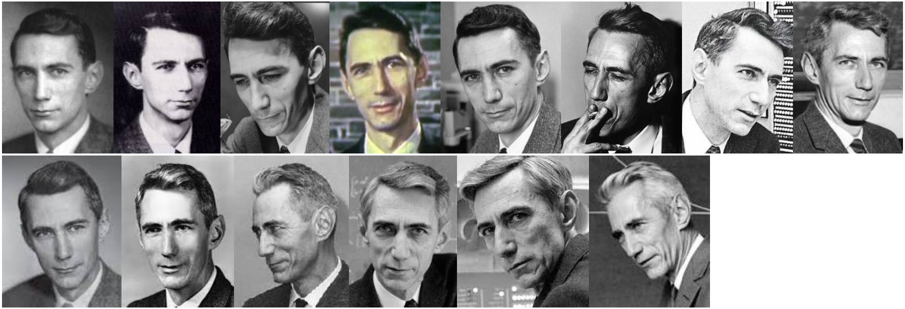

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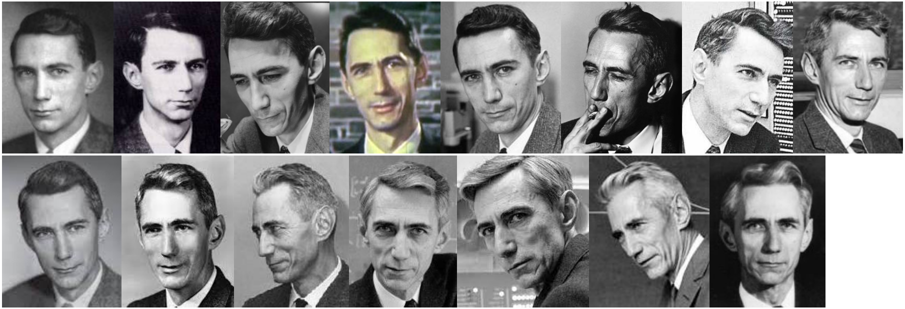

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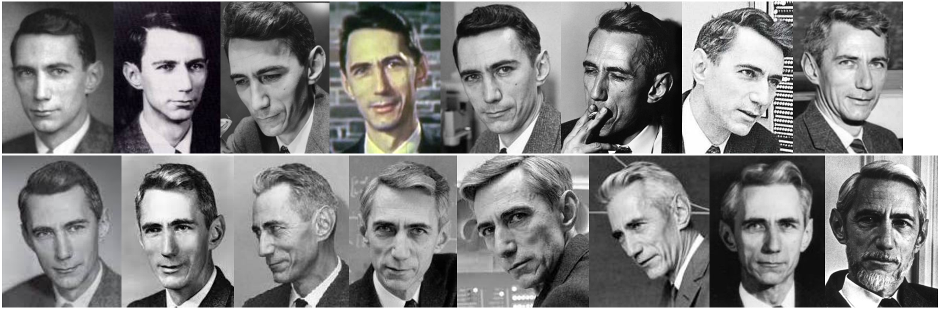

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"the most important man... you've never heard of"

Well-Known Scientific Heroes

Alan Turing (1912–1954)

Well-Known Scientific Heroes

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Well-Known Scientific Heroes

John Nash (1928–2015)

Well-Known Scientific Heroes

John Nash (1928–2015)

The Quiet and Modest Life of Shannon

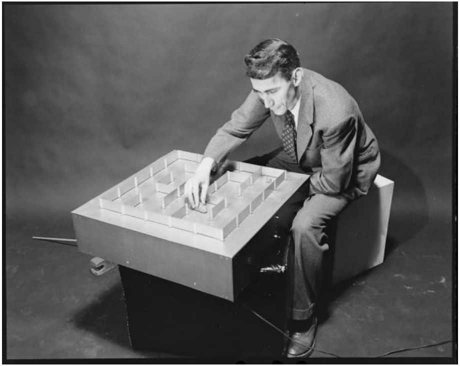
Shannon with Juggling Props

The Quiet and Modest Life of Shannon

Shannon's Toys Room

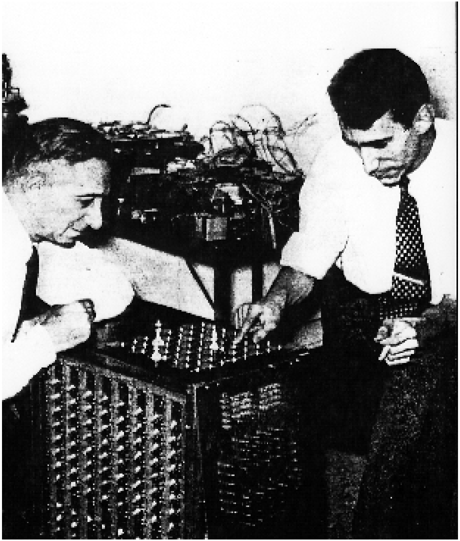
Shannon is known for riding through the halls of Bell Labs on a unicycle while simultaneously juggling four balls

Crazy Machines



Theseus (labyrinth mouse)

Crazy Machines



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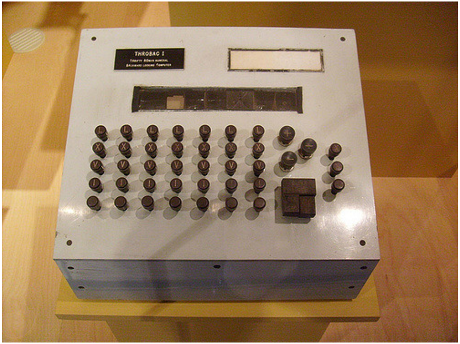
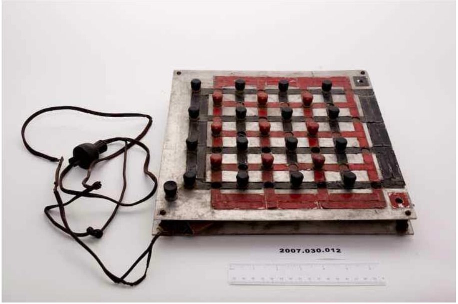


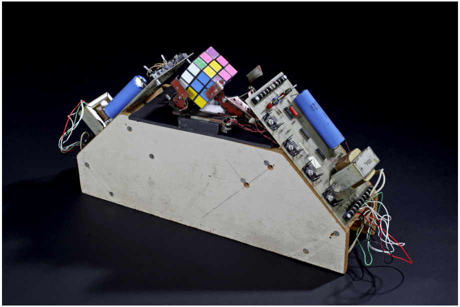
calculator in Roman numerals

Crazy Machines



“Hex” switching game machine

Crazy Machines

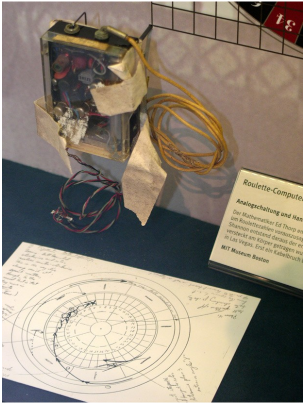


Rubik's cube solver

Crazy Machines

3-ball juggling machine

Crazy Machines



Wearable computer to predict roulette in casinos
(with Edward Thorp)

Crazy Machines

ultimate useless machine

“Serious” Work

At the same time, Shannon made decisive theoretical advances in ...

- logic & circuits
- cryptography
- artifical intelligence
- stock investment
- wearable computing

⋮

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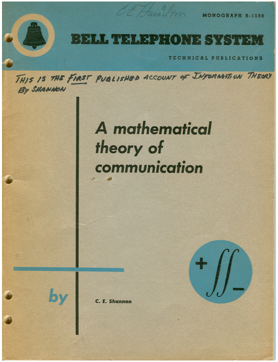
  ⋮

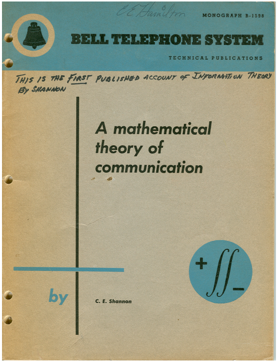
- ...and **information theory**!

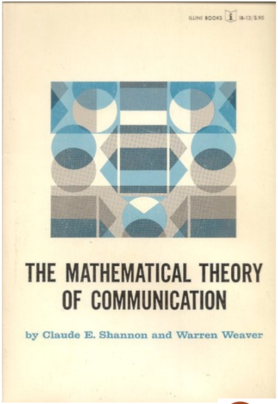
Outline

The Mathematical Theory of Communication (BSTJ, 1948)



The Mathematical Theory of Communication (BSTJ, 1948)





One article (written 1940–48): **A REVOLUTION !!!!!!**

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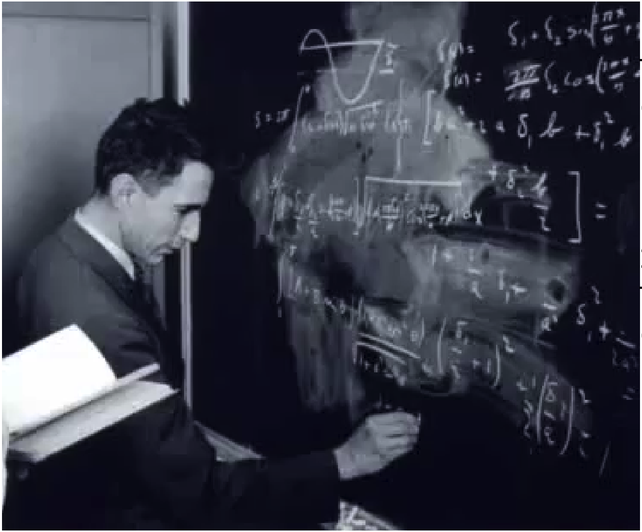
Shannon’s Theorems

Yes it’s Maths !!



1. Source Coding Theorem (*Compression* of Information)

2. Channel Coding Theorem (*Transmission* of Information)

Outline

Shannon's Paradigm

34 *The Mathematical Theory of Communication*

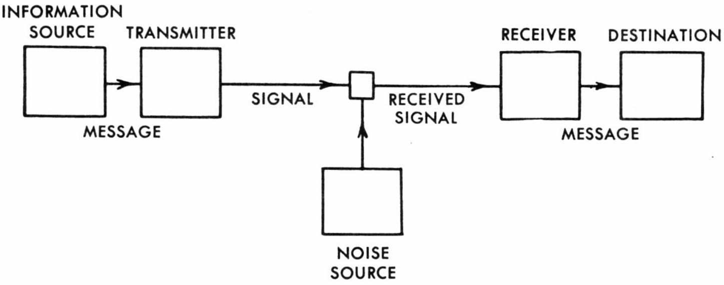
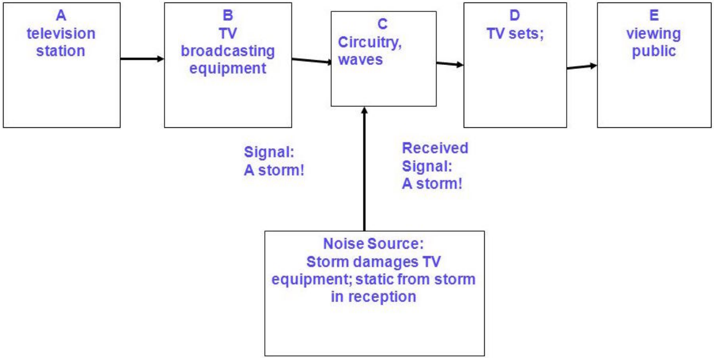


Fig. 1. — Schematic diagram of a general communication system.

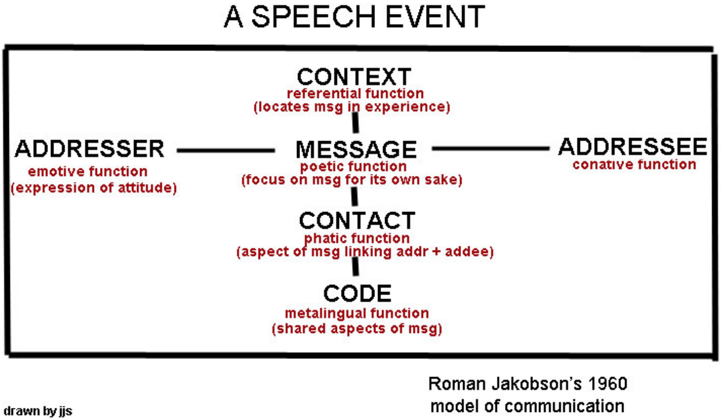
A tremendous impact!

Shannon's Paradigm... in Communication

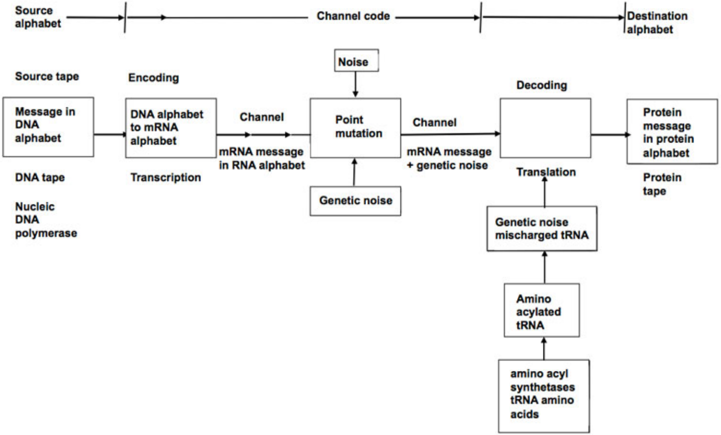
Example: Broadcast following crisis



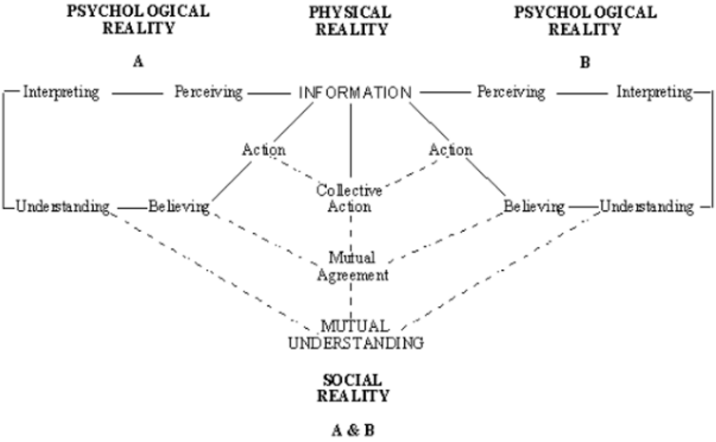
Shannon's Paradigm... in Linguistics



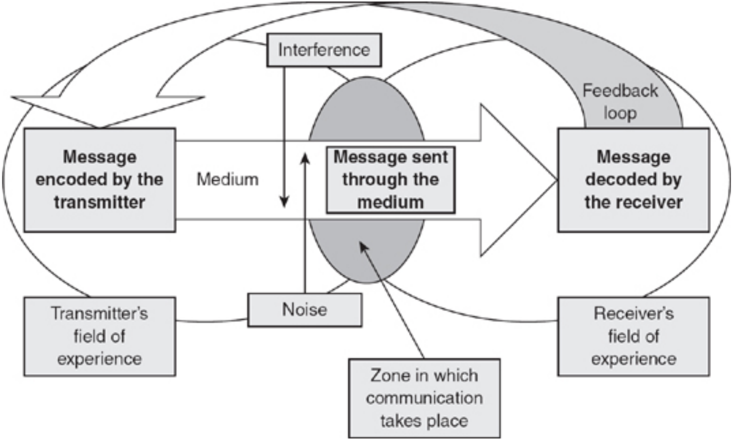
Shannon's Paradigm... in Biology



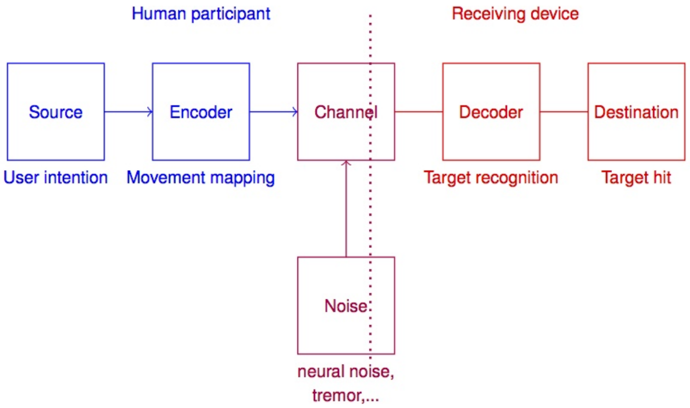
Shannon's Paradigm... in Psychology



Shannon’s Paradigm... in Social Sciences



Shannon's Paradigm... in Human-Computer Interaction



Shannon's "Bandwagon" Editorial

The Bandwagon

CLAUDE E. SHANNON

INFORMATION theory has, in the last few years, become something of a scientific bandwagon. Starting as a technical tool for the communication engineer, it has received an extraordinary amount of publicity in the popular as well as the scientific press. In part, this has been due to connections with such fashionable fields as computing machines, cybernetics, and automation; and in part, to the novelty of its subject matter. As a consequence, it has perhaps been ballooned to an importance beyond its actual accomplishments. Our fellow scien-

subject are aimed in a very specific direction, a direction that is not necessarily relevant to such fields as psychology, economics, and other social sciences. Indeed, the hard core of information theory is, essentially, a branch of mathematics, a strictly deductive system. A thorough understanding of the mathematical foundation and its communication application is surely a prerequisite to other applications. I personally believe that many of the concepts of information theory will prove useful in these other fields—and, indeed, some results are already quite

Shannon's Viewpoint

> *"The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.*
>
> *Frequently the messages have meaning; [...] These semantic aspects of communication are irrelevant to the engineering problem.*
>
> *The significant aspect is that the actual message is one selected from a set of possible messages [...] unknown at the time of design. "*

X : a message symbol modeled as a **random variable**

$p(x)$: the **probability** that $X = x$

Kolmogorov's Modern Probability Theory

Andreï Kolmogorov (1903–1987)

- founded modern probability theory in 1933
- a strong early supporter of information theory!

"Information theory must precede probability theory and not be based on it. [...] The concepts of information theory as applied to infinite sequences [...] can acquire a certain value in the investigation of the algorithmic side of mathematics as a whole."

A Logarithmic Measure

- 1 digit represents 10 numbers 0,1,2,3,4,5,6,7,8,9;
- 2 digits represents 100 numbers 00, 01, . . . , 99;
- 3 digits represents 1000 numbers 000, . . . , 999;

  ⋮
- $\log_{10} M$ digits represents M possible outcomes

Ralph Hartley (1888–1970)

"[...] take as our practical measure of information the logarithm of the number of possible symbol sequences"

Transmission of Information, BSTJ, 1928

The Bit

- $\log_{10} M$ digits represents M possible outcomes
- or...
- $\log_2 M$ **bits** represents M possible outcomes

 John Tukey (1915–2000)

coined the term "bit" (contraction of "binary digit")
which was first used by Shannon in his 1948 paper

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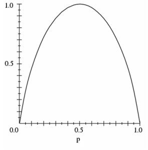
any information can be represented by a sequence of 0’s and 1’s — the Digital Revolution!

The Unit of Information

bit (binary digit, unit of storage) $\neq$ bit (binary unit of information)

- less-likely messages are more informative than more-likely ones
- 1 bit is the information content of one equiprobable bit $(\frac{1}{2}, \frac{1}{2})$

  otherwise the information content is < 1 bit:



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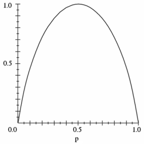

The official name (International standard ISO/IEC 80000-13) for the information unit:

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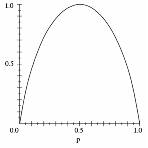
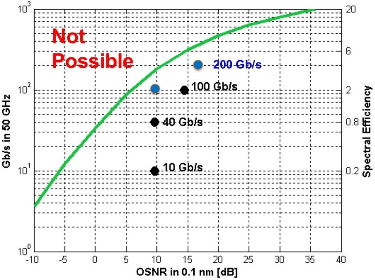


The official name (International standard ISO/IEC 80000-13) for the information unit:

...the **Shannon** (symbol Sh)

Fundamental Limit of Performance

- Shannon does not really give *practical* solutions but solves a *theoretical* problem:
- *No matter what you do*,
  (as long as you have a given amount of ressources)
  you *cannot* go beyond than a certain bit rate limit
  to achieve reliable communication



Fundamental Limit of Performance

- before Shannon: 
  communication technologies did *not* have a landmark
- the limit can be calculated: we know how far we are from it and you can be (in theory) *arbitrarily close* to the limit!
- the challenge becomes:
  how can we build practical solutions that are close to the limit?

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- the challenge becomes:
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Asymptotic Results

- to find the limits of performance, Shannon's results are necessarily asymptotic
- a source is modeled as a sequence of random variables

$$X_1, X_2, \ldots, X_n$$

  where the dimension $n \to +\infty$.
- this allows to exploit dependences
  and obtain a geometric "gain"
  using the *law of large numbers*
  where limits are expressed as *expectations* $\mathbb{E}\{\cdot\}$

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Asymptotic Results: Example

Consider the source $X_1, X_2, \ldots, X_n$ where each X can take a finite number of possible values, independently of the other symbols.

The probability of message $\underline{x} = (x_1, x_2, \ldots, x_n)$ is the product of the individual probabilities:

$$p(\underline{x}) = p(x_1) \cdot p(x_2) \cdot \cdots \cdot p(x_n).$$

Re-arrange according to the value x taken by each argument:

$$p(\underline{x}) = \prod_x p(x)^{n(x)}$$

where $n(x) =$ number of symbols equal to x.

Asymptotic Results: Example (Cont'd)

By the *law of large numbers*, the empirical probability (frequency)

$$\frac{n(x)}{n} \to p(x) \qquad \text{as } n \to +\infty$$

Therefore, a "typical" message $\underline{x} = (x_1, x_2, \ldots, x_n)$ satisfies

$$p(\underline{x}) = \prod_x p(x)^{n(x)} \approx \prod_x p(x)^{np(x)}$$

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$$p(\underline{x}) = \prod_x p(x)^{n(x)} \approx \prod_x p(x)^{np(x)} = 2^{-n \cdot H}$$

where

$$\boxed{H = \sum_x p(x) \log_2 \frac{1}{p(x)}} = \mathbb{E}\left\{\log_2 \frac{1}{p(X)}\right\}$$

is a positive quantity called entropy.

Shannon's entropy $H = \sum_x p(x) \log_2 \frac{1}{p(x)}$

- analogy with statistical mechanics

Ludwig Boltzmann (1844–1906)

Shannon's entropy

$$H = \sum_x p(x) \log_2 \frac{1}{p(x)}$$

- analogy with statistical mechanics

Ludwig Boltzmann (1844–1906)

- suggested by

John von Neumann (1903–1957)

Shannon's entropy

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Ludwig Boltzmann (1844–1906)

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"You should call it entropy [...] no one really knows what entropy really is, so in a debate you will always have the advantage."

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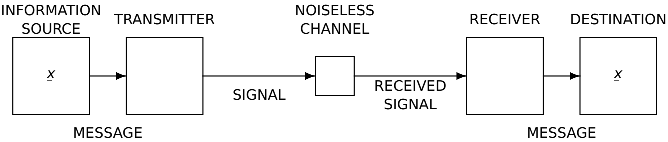
John von Neumann (1903–1957)

- studied in physics by

Léon Brillouin (1889–1969)

The Source Coding Theorem

Compression problem: noiseless channel, minimize bit rate



A "typical" sequence $\underline{x} = (x_1, x_2, \ldots, x_n)$ satisfies $p(\underline{x}) \approx 2^{-nH}$.
Summing over the N typical sequences:

$$1 \approx N\,2^{-nH}$$

since the probability of $\underline{x}$ being typical is ≈ 1. So $N \approx 2^{nH}$.
It is sufficient to encode only the N typical sequences:

$$\frac{\log_2 N}{n} \approx H \quad \text{bits per symbol}$$

The Source Coding Theorem

Theorem (Shannon's First Theorem)

Only H bits per symbol suffice to reliably encode an information source.

The entropy H is the bit rate lower bound for reliable compression.

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Theorem (Shannon's First Theorem)

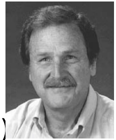
Only H bits per symbol suffice to reliably encode an information source.

The entropy H is the bit rate lower bound for reliable compression.

- This is an asymptotic theorem ($n \to +\infty$) not a practical solution.
- Variable length coding solution by Shannon and

Robert Fano (1917–2016)

- Optimal code (1952) by David Huffman (1925-1999)



- Elias, Golomb, Lempel-Ziv, ...

Relative Entropy (or Divergence)

$$\boxed{D(p,q) = \sum_x p(x) \log_2 \frac{p(x)}{q(x)}} \geq 0 \text{ with } D(p,q) = 0 \text{ iff } p \equiv q.$$

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Bounds of the type $2^{-n \cdot D(p,q)}$ useful in statistics:

- large deviations theory
- asymptotic behavior in hypothesis testing

Chernoff information to classify empirical data

Herman Chernoff (1923–)

Fisher information for parameter estimation

- Ronald Fisher (1890–1962)

Shannon's Mutual Information

Shannon's entropy of a random variable X:

$$H(X) = \sum_x p(x) \log_2 \frac{1}{p(x)} = \mathbb{E}\left\{\log_2 \frac{1}{p(X)}\right\}$$

Shannon's (mutual) information between two random variables X, Y:

$$I(X;Y) = \sum_{x,y} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)} = \mathbb{E}\left\{\log_2 \frac{p(X,Y)}{p(X)p(Y)}\right\}$$

This exactly $D(p,q)$ where:

- $p(x,y)$ is the (true) joint distribution;
- $q(x,y) = p(x)p(y)$ is what would have been in the case of *independence*.

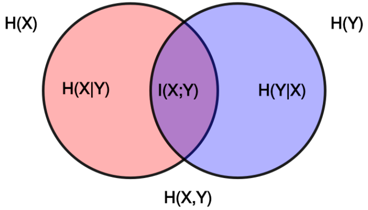
Therefore $I(X;Y) \geq 0$ with $I(X;Y) = 0$ iff X and Y are independent.

Shannon's Mutual Information

Shannon writes

$$I(X;Y) = \mathbb{E}\left\{\log_2 \frac{p(X|Y)}{p(X)}\right\} = H(X) - H(X|Y)$$

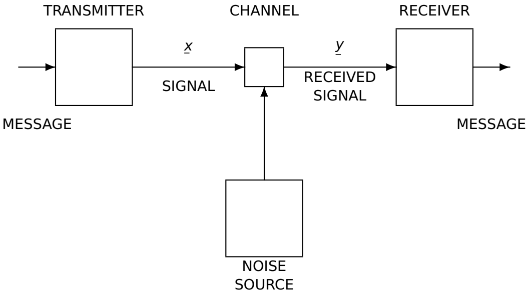
where $H(X|Y)$ is the conditional entropy of X given Y.



- $H(X|Y) \leq H(X)$: *knowledge decreases uncertainty* by a quantity equal to the information gain $I(X;Y)$.
- intuitive and rigorous!

The Channel Coding Theorem

Transmission problem: noisy channel, maximize bit rate for reliable communication



It is sufficient to decode only sequences $\underline{x}$ jointly typical with $\underline{y}$.

The Channel Coding Theorem (Cont'd)

But another code is also jointly typical with $\underline{y}$ with probability bounded by

$$2^{-n \cdot I(X;Y)}.$$

Summing over the N code sequences, the total probability of decoding error is bounded by

$$N \cdot 2^{-n \cdot I(X;Y)}$$

which tends to zero only if the bit rate

$$\frac{\log_2 N}{n} < I(X;Y)$$

Definition (Channel Capacity)

$$C = \max_{p(x)} I(X;Y)$$

The Channel Coding Theorem (Cont'd)

If the bit rate is $< C$, then the error probability, averaged over all possible codes, can be made as small as desired.

Therefore there exists at least one code with arbitrarily small probability of error.

Theorem (Shannon's Second Theorem)

Information can be transmitted reliably provided that the bit rate does not exceed the channel capacity C.

The capacity C is the bit rate upper bound for reliable transmission.

The Channel Coding Theorem (Cont'd)

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The capacity C is the bit rate upper bound for reliable transmission.

Revolutionary! Transmission noise does not affect quality—it only impacts the bit rate.
This is the theorem that led to the digital revolution!

Shannon's Result is Paradoxical!

- Shannon theorems show that good codes exist, but give no clue on how to build them in practice
- but choosing a code at random would be almost optimal!
- however random coding is impractical (n is large)...
- only 50 years later were found *turbo-codes* (by Claude Berrou & Alain Glavieux) that imitate random coding to approach capacity

Outline

c l a u d e   s h a n n o n

anagram

claude shannon

a sound channel

c l a u d e   s h a n n o n

a sound channel

Shannon's formula:

$$C = W \log_2\left(1 + \frac{P}{N}\right) \quad \text{bits/second}$$

claude shannon

a sound channel

Shannon's formula:

or... $$C = \frac{1}{2}\log_2\left(1+\frac{P}{N}\right)$$ bits/symbol

Additive White Gaussian Noise Channel

A very common model: $Y = X + Z$ where Z is Gaussian $\mathcal{N}(0, \sigma^2)$.

Shannon finds the exact expression:

$$\boxed{C = W \cdot \log_2\left(1 + \frac{P}{N}\right)} \quad \text{bit/s}$$

where W is the bandwidth and P/N is the signal-to-noise ratio.

- a "concrete" finding of information theory – the most celebrated formula of Shannon!
- to derive this formula, Shannon popularized the Whittaker-Nyquist sampling theorem — "Shannon's Theorem"!

Claude Shannon

Shannon's formula:

$$C = W \log_2 \left(\frac{P+N}{N} \right)$$

"A Mathematical Theory of Communication," *The Bell System Technical Journal*, Vol. 27, pp. 623–656, October, 1948.

In the end, "The Mathematical Theory of Communication," [1] and the book based on it [25] came as a bomb, and something of a delayed-action bomb.

Claude Shannon

Shannon's formula:

$$C = W \log_2\left(\frac{P+N}{N}\right)$$

"A Mathematical Theory of Communication," *The Bell System Technical Journal*, Vol. 27, pp. 623–656, October, 1948.

Note on the Theoretical Efficiency of Information Reception with PPM*

For small P/N ratios, the now classical expression for the information reception capacity of a channel

$$C = W \lg_2 (1 + P/N)$$

can be written, substituting kTW for N,

$$CT_0 = WT\, P/N \lg_2 e = \frac{PT_0}{kT} \lg_2 e = \frac{E}{kT} \lg_2 e$$

* Received by the Institute, February 23, 1949.

Ralph Hartley

20 years before... in the same journal...

Hartley's rule:

$$\boxed{C' = \log_2\left(1 + \frac{A}{\Delta}\right)} \quad \text{bits/symbol}$$

"Transmission of Information," *The Bell System Technical Journal*, Vol. 7, pp. 535–563, July 1928 .

Ralph Hartley

Hartley's rule:

$$C' = \log_2\left(1 + \frac{A}{\Delta}\right)$$

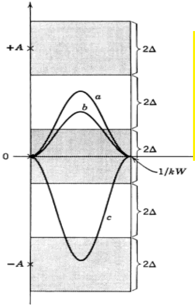

Figure 1.1 Distinguishable receiver amplitudes. Hartley considered received pulse amplitudes to be distinguishable only if they lie in different zones of width 2Δ. Thus pulses a and c are distinguishable but a and b are not. For the case shown, $A/\Delta = 4$ and there are five distinguishable zones.

(Wozencraft-Jacobs textbook, 1965)

Ralph Hartley

Hartley's rule:

$$C' = \log_2\left(1 + \frac{A}{\Delta}\right)$$

- amplitude "SNR" A/Δ (factor 1/2 is missing)
- no coding involved (except quantization)
- zero error

Hartley's formulation exhibits a simple but somewhat inexact interrelation among the time interval T, the channel bandwidth W, the maximum signal magnitude A, the receiver accuracy Δ, and the allowable number M of message alternatives. Communication theory is intimately concerned with the determination of more precise interrelations of this sort.

(Wozencraft-Jacobs textbook, 1965)

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Outline

Hartley's $C' = \log_2\left(1 + \frac{A}{\Delta}\right)$ came 20 years before Shannon

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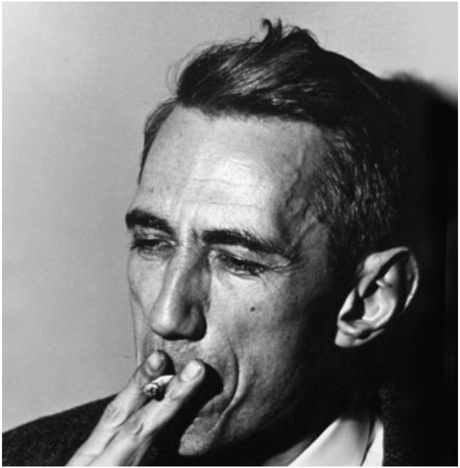
Hartley's rule is inexact: $C' \neq C$

Besides, C' is not the capacity of a noisy channel

Wrong!



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Outline

- Who is Claude Shannon?
- Shannon's Seminal Paper
- Shannon's Main Contributions
- Shannon's Capacity Formula
- Hartley's rule $C' = \log_2\left(1 + \frac{A}{\Delta}\right)$ is not Hartley's
- Many authors independently derived $C = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right)$ in 1948.
- Hartley's rule is exact: $C' = C$ (a coincidence?)
- C' is the capacity of the "uniform" channel
- Shannon's Conclusion

Hartley or not Hartley

Quote from Shannon, 1984:

> aspects of information theory. I started with information theory, inspired by Hartley's paper, which was a good paper, but it did not take account of things like noise and best encoding and probabilistic aspects.[3]

- In Hartley's paper, no mention of signal vs. noise or A vs. Δ
- Why was $C' = \log_2\left(1 + \frac{A}{\Delta}\right)$ mistakenly attributed to Hartley?

The first tutorial of information theory!

A HISTORY OF THE THEORY OF INFORMATION

By E. COLIN CHERRY, M.Sc., Associate Member.

(The paper was first received 7th February, and in revised form 28th May, 1951.)

⋮

increased. Although not explicitly stated in this form in his paper, Hartley[12] has implied that the quantity of information

⋮

distinguishable" amplitude change; in practice this smallest step may be taken to equal the *noise level*, *n*. Then the quantity of information transmitted may be shown to be proportional to

$$Bt \log\left(1 + \frac{a}{n}\right)$$

Outline

And then there were eight

Quote from Shannon, 1948:

Formulas similar to $C = W \log \frac{P+N}{N}$ for the white noise case have been developed independently by several other writers, although with somewhat different interpretations. We may mention the work of N. Wiener,[7] W. G. Tuller,[8] and H. Sullivan in this connection.

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6. André G. Clavier, December 1948
7. Stanford Goldman, May 1948
8. Claude E. Shannon, Oct. 1948

Norbert Wiener

III

Time Series, Information, and Communication

There is a large class of phenomena in which what is observed is a numerical quantity, or a sequence of numerical quantities, dis-

An interesting problem is that of determining the information gained by fixing one or more variables in a problem. For example, let us suppose that a variable u lies between x and $x + dx$ with the probability $\exp(-x^2/2a)\, dx/\sqrt{2\pi a}$, while a variable v lies between the same two limits with a probability $\exp(-x^2/2b)\, dx/\sqrt{2\pi b}$. How much information do we gain concerning u if we know that $u + v = w$? In this case, it is clear that $u = w - v$, where w is

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Norbert Wiener

The excess of information concerning x when we know w to be that which we have in advance is

$$\frac{1}{\sqrt{2\pi[ab/(a+b)]}}\int_{-\infty}^{\infty}\left\{\exp\left[-(x-c_2)^2\left(\frac{a+b}{2ab}\right)\right]\right\}$$
$$\times\left[-\frac{1}{2}\log_2 2\pi\left(\frac{ab}{a+b}\right)\right]-(x-c_2)^2\left[\left(\frac{a+b}{2ab}\right)\right]\log_2 e\Big]\,dx$$
$$-\frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty}\left[\exp\left(-\frac{x^2}{2a}\right)\right]\left(-\frac{1}{2}\log_2 2\pi a-\frac{x^2}{2a}\log_2 e\right)dx$$
$$=\frac{1}{2}\log_2\left(\frac{a+b}{b}\right) \qquad (3.091)$$

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$$\times\left[-\frac{1}{2}\log_2 2\pi\left(\frac{ab}{a+b}\right)\right] - (x-c_2)^2\left[\left(\frac{a+b}{2ab}\right)\right]\log_2 e\Big]\,dx$$
$$-\frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty}\left[\exp\left(-\frac{x^2}{2a}\right)\right]\left(-\frac{1}{2}\log_2 2\pi a-\frac{x^2}{2a}\log_2 e\right)dx$$
$$=\frac{1}{2}\log_2\left(\frac{a+b}{b}\right) \qquad (3.091)$$

Later. . . in 1956:

What is Information Theory?

NORBERT WIENER

INFORMATION THEORY has been identified in the public mind to denote the theory of information by bits, as developed by Claude E. Shannon and myself. This notion is certainly impor-

Charles W. Earp

Relationship Between Rate of Transmission of Information, Frequency Bandwidth, and Signal-to-Noise Ratio*

By C. W. EARP

Standard Telephones and Cables, Limited, London, England

nels, channel maximum signal to root-mean-square noise ratio $= S_{\mathrm{SSB}}/\sqrt{n}$ and maximum signal-to-*peak*-noise ratio $= S_{\mathrm{SSB}}/(p\sqrt{n})$.

In each channel, the available power may be used to provide N instantaneous values, this being achieved without ambiguity provided that

$$N < \left(\frac{S_{\mathrm{SSB}}}{p\sqrt{n}} + 1\right).$$

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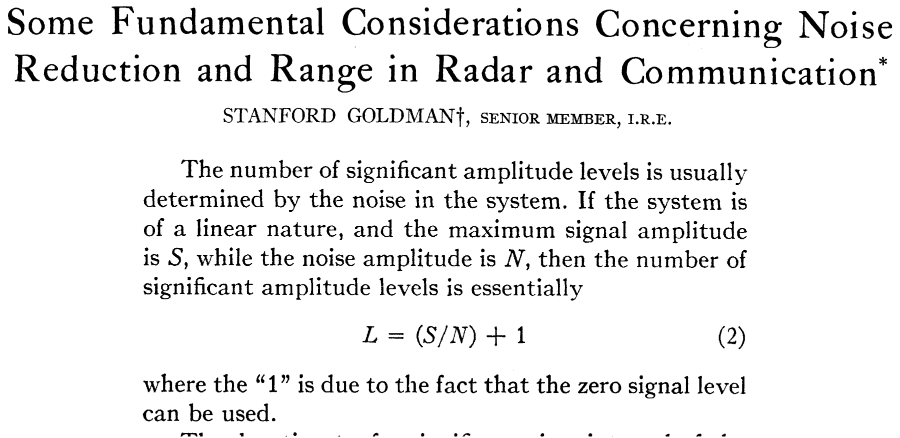
* The present paper was written in original form in October, 1946, when the author had no knowledge of any practical development of pulse-code modulation, as the

Stanford Goldman


Some Fundamental Considerations Concerning Noise Reduction and Range in Radar and Communication*

STANFORD GOLDMAN†, SENIOR MEMBER, I.R.E.

The number of significant amplitude levels is usually determined by the noise in the system. If the system is of a linear nature, and the maximum signal amplitude is S, while the noise amplitude is N, then the number of significant amplitude levels is essentially

$$L = (S/N) + 1 \tag{2}$$

where the "1" is due to the fact that the zero signal level can be used.

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[4] Equation (5) has been derived independently by many people, among them W. G. Tuller, from whom the writer first learned about it.

William G. Tuller

Theoretical Limitations on the Rate of Transmission of Information*

WILLIAM G. TULLER†, SENIOR MEMBER, IRE

recognizable.[14] Then, if N is the rms amplitude of the noise mixed with the signal, there are $1+S/N$ significant values of signal that may be determined. This sets s in

⋮

have from (1) the quantity of information available at the output of the system:

$$H = kn \log s = k2f_cT \log (1 + S/N). \qquad (2)$$

This is an important expression, to be sure, but gives

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[10] C. E. Shannon, "Communication in the presence of noise," Proc. I.R.E., vol. 37, pp. 10–22; January, 1949.

[11] The existence of this work was learned by the author in the spring of 1946, when the basic work underlying this paper had just been completed. Details were not known by the author until the summer of 1948, at which time the work reported here had been complete for about eight months.

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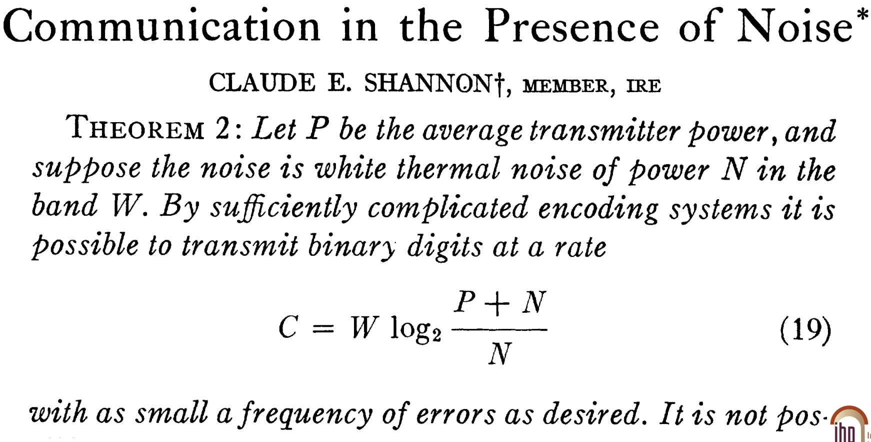
Claude E. Shannon



Communication in the Presence of Noise*

CLAUDE E. SHANNON†, MEMBER, IRE

THEOREM 2: *Let P be the average transmitter power, and suppose the noise is white thermal noise of power N in the band W. By sufficiently complicated encoding systems it is possible to transmit binary digits at a rate*

$$C = W \log_2 \frac{P+N}{N} \qquad (19)$$

with as small a frequency of errors as desired. It is not pos-

Claude E. Shannon

Communication in the Presence of Noise*

CLAUDE E. SHANNON†, MEMBER, IRE

* Decimal classification: 621.38. Original manuscript received by the Institute, July 23, 1940. Presented, 1948 IRE National Convention, New York, N. Y., March 24, 1948; and IRE New York Section, New York, N. Y., November 12, 1947.

Communication in the Presence of Noise*

CLAUDE E. SHANNON†, MEMBER, IRE

[10] A. Hodges, *Alan Turing: The Enigma*, New York: Simon and Schuster, 1983. [The following information was obtained from C. E. Shannon on March 3, 1984: "On p. 552, Hodges cites a Shannon manuscript date of 1940, which is, in fact, a typographical error. While results for coding statistical sources into noiseless channels using the *p*log(*p*) measure were obtained in 1940–1941 (at the Institute for Advanced Study in Princeton), first submission of this work for formal publication occurred soon after World War II."]

What about the French?

Deux ingénieurs français ont publié la même « formule de Shannon » en 1948:

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Clavier & Laplume

André G. Clavier

Evaluation of Transmission Efficiency According to Hartley's Expression of Information Content*

By A. G. CLAVIER

Federal Telecommunication Laboratories, Incorporated, Nutley, New Jersey

small percentage of error due to noise. The total number of distinguishable levels on the ideal

line is thus given by

$$\frac{S+\bar{N}\sqrt{2}}{\bar{N}\sqrt{2}}=1+\frac{S}{\bar{N}\sqrt{2}},$$

with a reasonable approximation. It follows that the amount of information transmittible on the ideal line is measured by

$$H_{lm}=k_0\cdot 2f_l\cdot t\cdot \log\left(1+\frac{S_l}{\bar{N}_l\sqrt{2}}\right).$$

André G. Clavier

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* A symposium on "Recent Advances in the Theory of Communication" was presented at the November 12, 1947, meeting of the New York Section of the Institute of Radio Engineers. Four papers were presented by A. G. Clavier, Federal Telecommunication Laboratories; B. D. Loughlin, Hazeltine Electronics Corporation; and J. R. Pierce and C. E. Shannon, both of Bell Telephone Laboratories. The

Jacques Laplume

Meanwhile (1948), far away. . .

PHYSIQUE MATHÉMATIQUE. — *Sur le nombre de signaux discernables en présence du bruit erratique dans un système de transmission à bande passante limitée.* Note de M. JACQUES LAPLUME.

Si N et n sont suffisamment grands, on peut former une expression approchée de $\log M$ en utilisant la formule de Stirling limitée aux termes prépondérants. On trouve ainsi

$$\log M \simeq N \log \frac{N+n}{N} + n \log \frac{N+n}{n}. \tag{2}$$

Si, de plus, $N \gg n$,

$$\log M \simeq n \log \frac{N}{n} = TW \log \frac{P}{b}. \tag{3}$$

More on Jacques Laplume...

INSTITUT DE FRANCE
Académie des sciences

Histoire des sciences / Évolution des disciplines et histoire des découvertes — Octobre 2016

Laplume, sous le masque

par Patrick Flandrin (directeur de recherche CNRS à l'École normale supérieure de Lyon, membre de l'Académie des sciences) et Olivier Rioul (professeur à Télécom-ParisTech et professeur chargé de cours à l'École Polytechnique)

Cette note vise à faire sortir de l'oubli un travail original de 1948 de l'ingénieur français Jacques Laplume, relatif au calcul de la capacité d'un canal bruité de bande passante donnée. La publication de sa Note dans les Comptes Rendus de l'Académie des sciences a précédé de peu celle de l'article du mathématicien américain Claude E. Shannon, fondateur de la théorie de l'information, ainsi que celles de plusieurs chercheurs aux U.S.A.

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Who's formula?

The "Shannon-Hartley" formula

$$\boxed{C = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right)}$$

Who’s formula?

The “Shannon-Hartley” formula

$$\boxed{C = \frac{1}{2}\log_2\left(1+\frac{P}{N}\right)}$$

would actually be the

Shannon-Laplume-Tuller-Wiener-Clavier-Earp-Goldman-Sullivan formula

Outline

"Hartley"'s argument

The channel input X is taking $M = 1 + A/\Delta$ equiprobable values in the set $\{-A, -A + 2\Delta, \ldots, A - 2\Delta, A\}$:

$$P = \mathbb{E}(X^2) = \frac{1}{M}\sum_{k=0}^{n}(M - 1 - 2k)^2 = \Delta^2 \frac{M^2 - 1}{3}.$$

The input is mixed with additive noise Z with accuracy $\pm\Delta$, i.e. having uniform distribution in $[-\Delta, \Delta]$:

$$N = \mathbb{E}(Z^2) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} z^2 \mathrm{d}z = \frac{\Delta^2}{3}.$$

“Hartley”’s argument

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$$N = \mathbb{E}(Z^2) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} z^2 \mathrm{d}z = \frac{\Delta^2}{3}.$$

Hence

$$\log_2\left(1+\frac{A}{\Delta}\right) = \tfrac{1}{2}\log_2(1+M^2-1) = \tfrac{1}{2}\log_2\left(1+\frac{3P}{\Delta^2}\right) = \tfrac{1}{2}\log_2\left(1+\frac{P}{N}\right)$$

i.e., $\boxed{C' = C}$. A mathematical coïncidence?

Outline

The uniform channel

The capacity of $Y = X + Z$ with additive *uniform* noise Z is

$$\begin{aligned}\max_{X \text{ s.t. } |X| \leq A} I(X;Y) &= \max_X h(Y) - h(Y|X)\\ &= \max_X h(Y) - h(Z)\\ &= \max_{X \text{ s.t. } |Y| \leq A+\Delta} h(Y) - \log_2(2\Delta)\end{aligned}$$

Choose X^* to be discrete uniform in $\{-A, -A+2\Delta, \ldots, A\}$, then $Y = X^* + Z$ has uniform density over $[-A-\Delta, A+\Delta]$, which maximizes differential entropy:

$$\begin{aligned}&= \log_2(2(A+\Delta)) - \log_2(2\Delta)\\ &= \boxed{\log_2\left(1 + \frac{A}{\Delta}\right)}\end{aligned}$$

What is the worst noise?

Thus $C' = \log_2\left(1 + \frac{A}{\Delta}\right)$ is *correct* as a capacity! But:

- the noise is *not* Gaussian, but uniform;
- signal limitation is *not* on the power, but on the amplitude.

What is the worst noise?

Thus $C' = \log_2\left(1 + \frac{A}{\Delta}\right)$ is *correct* as a capacity! But:

- the noise is *not* Gaussian, but uniform;
- signal limitation is *not* on the power, but on the amplitude.

Further analogy:

- Shannon used the entropy power inequality to show that under limited *power*, Gaussian is the worst possible noise in the channel:

$$\frac{1}{2}\log_2\left(1 + \alpha\frac{P}{N}\right) \leq C \leq \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right) + \frac{1}{2}\log_2\alpha,$$

  where $\alpha = N/\tilde{N} \geq 1$

- We can show: under limited *amplitude*, *uniform* noise is the worst possible noise one can inflict in the channel:

$$\log_2\left(1 + \frac{A}{\Delta}\right) \leq C' \leq \log_2\left(1 + \frac{A}{\Delta}\right) + \log_2\alpha,$$

Conclusion

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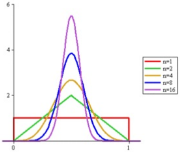
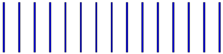
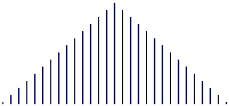
Rioul & Magossi, "On Shannon's formula and Hartley's rule: Beyond the mathematical coincidence," in Journal *Entropy*, Vol. 16, No. 9, pp. 4892-4910, Sept. 2014
http://www.mdpi.com/1099-4300/16/9/4892/

B-splines channels

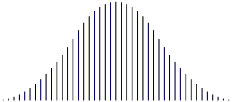




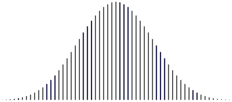
(a) $d = 0$ (rectangular)



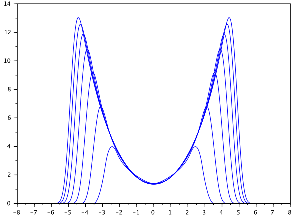
(b) $d = 1$ (triangular)



(c) $d = 2$



(d) $d = 3$

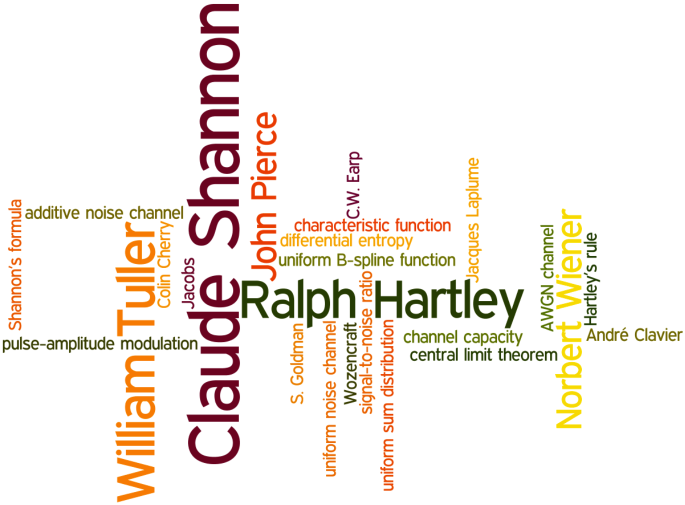


Outline

Shannon on Information Theory

"I didn't think at the first stages that it was going to have a great deal of impact. I enjoyed working on this kind of a problem, as I have enjoyed working on many other problems, without any notion of either financial or gain in the sense of being famous; and I think indeed that most scientists are oriented that way, that they are working because they like the game."

Thank you!


Claude Shannon
Ralph Hartley
William Tuller
John Pierce
Norbert Wiener
Shannon's formula
additive noise channel
Colin Cherry
Jacobs
pulse-amplitude modulation
characteristic function
differential entropy
uniform B-spline function
C.W. Earp
Jacques Laplume
S. Goldman
uniform noise channel
Wozencraft
signal-to-noise ratio
uniform sum distribution
channel capacity
central limit theorem
AWGN channel
Hartley's rule
André Clavier

Shannon's Formula $W \cdot \log(1 + SNR)$: A Historical Perspective

on the occasion of Shannon's Centenary

Oct. 26th, 2016

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