A Remez Exchange Algorithm for Orthonormal Wavelets

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***Abstract*—Compactly supported orthonormal wavelets are obtained from two-band paraunitary FIR filter bank solutions, with the additional "flatness" constraint that the low-pass filter should have K zeroes at half the sampling frequency. This constraint is set to obtain "regular" wavelets. However, it is somewhat in contradiction with the usual requirement for good frequency selectivity, since it is well known that maximally flat filters (yielding Daubechies wavelets) have poor frequency selectivity. An efficient procedure for designing maximally frequency selective filter banks under a given flatness constraint is described in this paper. Classical Remez exchange algorithms, based on the alternation theorem, can no longer be used in this case. Linear programming techniques are capable of setting up constraints of this type, but require high memory storage and computation time.**

First, a variation of the alternation theorem adapted to this new situation is derived. Then, a modified Remez exchange algorithm for the design of "wavelet" filters is derived in the spirit of the Parks-McClellan algorithm. The efficiency of the algorithm is greatly improved as compared to linear programming techniques, and optimum filters are generally obtained after 3 or 4 iterations. A MATLAB listing is provided.

I. Introduction

A. Wavelets and filter banks

The CONNECTION between continuous-time wavelets and discrete filter banks, originally investigated by Daubechies [1], is now well understood [11]. Compactly supported wavelets can be generated by perfect reconstruction two-band FIR filter bank solutions. In this paper, we consider a paraunitary filter bank [6], [12], [14], as depicted in Fig. 1, which, when iterated, generates orthonormal wavelets [1]. We also restrict ourselves to real-valued filters and wavelets.

Recall that a paraunitary solution pair $(G(z), H(z))$ of causal FIR filters of length L, where L is necessarily even, satisfies [6], [12] $G(z) = -z^{-(L-1)}H(-z^{-1})$. Therefore, only one filter, e.g. low-pass filter $H(z)$, has to be designed. The paraunitariness condition becomes [6], [12], [14]

$$P(z) + P(-z) = 2 \tag{1}$$

where the "product filter"

$$P(z) = H(z)H(z^{-1}) \tag{2}$$

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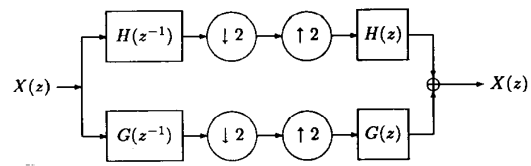

Fig. 1. Paraunitary two-band FIR filter bank (in non-causal form). $H(z)$ and $G(z)$ are half-band low-pass and high-pass filters, respectively.

is a zero-phase filter of length $2L-1$, whose impulse response is symmetric. In this paper, we concentrate on the design of $P(z)$ under magnitude specifications on $P(e^{j\omega}) = |H(e^{j\omega})|^2$. Clearly, conditions (1) and (2) are equivalent to requiring that $P(e^{j\omega})$ is a nonnegative trigonometric sum depending on $L/2$ variables,

$$P(e^{j\omega}) = 1 + \sum_{n=1}^{L/2} a_n \cos(2n-1)\omega \geq 0 \tag{3}$$

The actual solution $H(z)$ is then determined by factoring $P(z)$, as described in [1], [12] and Section V-A.

The connection between paraunitary solutions $(G(z), H(z))$ and wavelets can be described as follows. Suppose that the filter bank of Fig. 1 is iterated on the low-pass branch at each step of decomposition [12]. This generates equivalent band-pass filters of the form [9]

$$G^i(z) = H(z)\,H(z^2)\cdots H(z^{2^{i-2}})\,G(z^{2^{i-1}}). \tag{4}$$

Letting $i \to \infty$ gives what is known in wavelet theory as the "mother wavelet" $\psi(t)$ [1].

$$\psi(t) = \lim_{i\to\infty} g^i_{\lfloor t2^i \rfloor} \tag{5}$$

where g^i_n is the impulse response of $G^i(z)$.

B. Flatness condition

The filter bank is never iterated to infinity in practice, so the interest of computing the wavelet by (5) is limited. However, it was suggested in various works [11] that for some applications, a desirable property is the ability of the waveform of g^i_n to vary smoothly in time n. This can be imposed by requiring that the limit function $\psi(t)$ exists and is *regular* (i.e., continuous, possibly with several continuous derivatives [1]). Although regularity is mathematically stated as a property of $\psi(t)$, it can be characterized on the low-pass filter taps h_n (we refer the interested reader to [1,9] for further details).

Of course, frequency selectivity is also thought of as a useful property for many applications. But, as we shall see later, regularity and frequency selectivity somewhat contradict each other. For this reason, we found very useful to propose a procedure providing filters which, for a given regularity, have the best possible frequency selectivity. Therefore, regularity is here understood as a new filter design constraint.

Regularity is usually quantified by measuring the "regularity order," which can be defined as the number of times $\psi(t)$ is continuously differentiable. The regularity order of a wavelet generated by some filter h_n can be accurately estimated using algorithms given in [9], and optimal conditions for regularity can be found. However, these conditions could hardly be used as a design constraint. The simplest regularity condition for filter design is a *flatness* constraint on the magnitude response $|H(e^{j\omega})|$ at the Nyquist frequency ($\omega = \pi$): Kth-order flatness is obtained if $H(z)$ contains K zeroes located at $z = -1$. That is,

$$(1 + z^{-1})^{2K} \text{ divides } P(z) \tag{6}$$

(Note that from (1), the magnitude response is also flat at the zero frequency.) Highly regular wavelets require high values of K [1], [9]. Conversely, increasing K will generally increase regularity [1], although the effect on regularity of zeroes of $H(z)$ that are *not* located at $z = -1$ might sometimes inverse this tendency [2] (see also Section VII). However, we consider only (6) as a regularity condition in this paper.

C. Connection with previous related work

As shown in Section II–A, Kth order flatness reduces to K linear constraints on the $L/2$ variables a_n (3), leaving $L/2 - K$ degrees of freedom. The flatness constraint is, therefore, maximal for $K = L/2$. In this case, there remains no degree of freedom and closed-form expressions for $P(z)$ can be found in the literature [3], [5]. Solutions $P(z)$ are half-band maximally flat filters, as proposed by Herrmann [3]. The corresponding wavelets were proposed by Daubechies [1], hence solutions $(G(z), H(z))$ are often termed Daubechies filters of length $L = 2K$. Fig. 2 shows that these maximally flat solutions, generating regular wavelets, are poorly selective.

At the other extreme, if the flatness constraint is removed ($K = 0$), Mintzer [6] and Smith and Barnwell [12] showed that best frequency selective filters $P(z)$ can be designed using classical algorithms based on the alternation theorem, such as the well-known Parks-McClellan [8] and Hoffsetter-Oppenheim-Siegel [4] algorithms. In this case, $P(z)$ is a type-I extraripple filter [7] which, as a consequence of the alternation theorem, meets (6) for $K = 1$ if $L/2$ is odd. However, Fig. 2 shows that the corresponding wavelets are much less regular than Daubechies wavelets of the same length.

The aim of this paper is to provide an efficient wavelet filter design algorithm, which provides the best frequency selective filter $P(z)$ under a given flatness constraint (6), including the two extreme choices $K = 0$ and $K = L/2$. This will overcome the present limitation of available "orthonormal wavelet filters" [1], by providing a number of filters $(G(z), H(z))$ with balanced regularity, frequency selectivity, and number of taps.

It was pointed out to us by one of the reviewers that there is also a connection with [13], in which best frequency selective filters with flatness constraint can be designed using the Parks/McClellan algorithm [8]. Here, we take the additional paraunitary constraint of filter banks into account in order to provide "wavelet" solutions. As a result, as seen in Section IV, the Parks/McClellan algorithm can no longer be used.

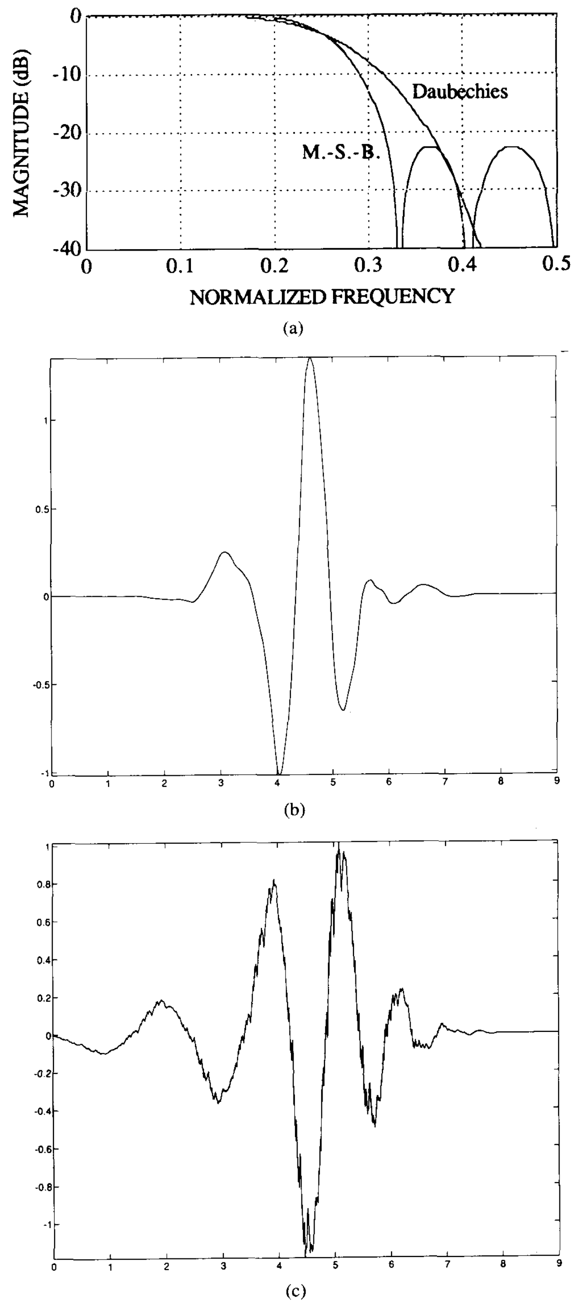

Fig. 2. Comparison between Daubechies and Mintzer-Smith-Barnwell (MSB) solutions ($L = 10$). (a). Magnitude responses of low-pass filters. The MSB solution was designed for a normalized transition bandwidth equal to 0.14. Corresponding Daubechies wavelet (b) and MSB wavelet (c).

This paper is organized as follows. Section II states the problem formally, while also pointing out the deficiencies

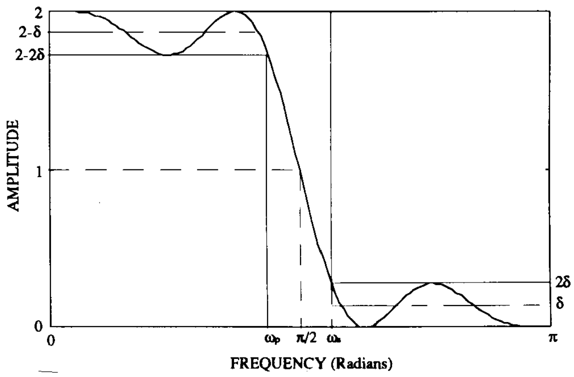

Fig. 3. Tolerance scheme for product filter $P(e^{j\omega})$. Note that positivity (3) requires a peak gain of 2 in the pass-band.

of linear programming techniques applied to our problem. Section III makes a change of variables to rewrite the problem in such a way that paraunitariness (1) *and* flatness (6) are built in the parameterization of $P(z)$. Section IV shows that although classical equiripple design algorithms can no longer be used, the alternation theorem still applies to our problem. Section V briefly describes the magnitude and phase characteristics of optimum filters. A modified Remez multiple exchange algorithm, adapted to our problem, is then derived in Section VI. Section VII shows that efficiency is greatly improved compared to linear programming techniques. Using a suitable initial guess of the extremal frequencies, the number of iterations needed in the algorithm seldom exceeds 3 for medium size filters $P(z)$ (of length ≤ 40).

II. Preliminaries

A. The Optimization Problem

The optimization problem addressed in this paper, briefly outlined in the introduction, is stated on the squared modulus of the frequency response of filter $H(z)$, $P(e^{j\omega}) = |H(e^{j\omega})|^2$: We would like to maximize stop-band attenuation of low-pass filter $P(z)$, where $P(e^{j\omega})$ is a nonnegative trigonometric polynomial of the form (3), while also requiring K-th order flatness (6). The tolerence scheme for this problem is depicted in Fig. 3.

Thanks to the built-in antisymmetry of $(P(e^{j\omega}) - 1)$ about $\omega = \pi/2$, all design specifications can be restricted to the first half-band $(0, \pi/2)$. In particular, the transition band (ω_p, ω_s) should be symmetric about $\omega = \pi/2$, i.e., $\omega_s = \pi - \omega_p$. Therefore, the optimization problem is stated as

$$\min_{a_n} \delta \tag{7}$$

under the inequality constraints

$$\sum_{n=1}^{L/2} a_n \cos(2n-1)\omega \leq 1, \;\; \omega \in [0, \pi/2], \tag{8}$$

$$\sum_{n=1}^{L/2} a_n \cos(2n-1)\omega \geq 1 - 2\delta, \;\; \omega \in [0, \omega_p], \tag{9}$$

and flatness constraints (6), which can be expressed as

$$\sum_{n=1}^{L/2} a_n = 1 \text{ (if } K > 0), \tag{10}$$

$$\sum_{n=1}^{L/2} a_n (2n-1)^{2k} = 0, \;\; k = 1, \ldots, K-1, \tag{11}$$

by requiring that the first $2K-1$ derivatives of $P(e^{j\omega})$ vanish at $\omega = 0$.

It is important to note that a solution to this optimization problem exists, because the set of filters $P(z)$ satisfying the constraints (8)–(11) is not empty: For any value of $K \leq L/2$, the maximally flat filter of length $4K - 1$ belongs to this set.

B. Linear Programming Techniques

Since positivity (8), frequency selectivity (9), and flatness (10), (11) are linear conditions in the coefficients a_n, linear programming techniques are capable of setting constraints of this type and provide optimal solutions. The linear program (7)–(11) is easily solved, for the $L/2 + 1$ unknowns δ and a_n, using any linear programming routine.

In Section III, we show that the deviation of the optimal solution should be maximal at the cut-off frequencies ω_p and ω_s, hence, $P(e^{j\omega_p}) = 2 - 2\delta$. Therefore, an alternative is to rewrite the linear program in the $L/2$ variables a_n. Another alternative is to first reduce the number of variables using the equality constraints (10) and (11), then work with the remaining $L/2 - K$ variables.

In the procedure, ω should be discretized in a fine grid in the interval $(0, \pi/2)$. The number of grid points is typically set to $16(L/2 - K)$. Grids should include the extremum frequency ω_p but should exclude $\omega = 0$ for $K > 0$ to avoid conditioning problems. Also, for increasing values of L and K, the matrix of the linear system (10), (11) soon becomes badly conditioned, causing convergence problems. The conditioning can be greatly improved by taking suitable linear combinations of (10), (11), yielding K flatness constraints of the general form $\sum_n a_n Q_k(2n-1) = Q_k(0)$, where the $Q_k(x)$'s are K linearly independent polynomials in x^2 of degree $\leq K - 1$.

Despite all the care that can be taken to write the linear program, the resulting procedure is not particularly adapted to the specificity of our filter design problem. As a result, compared to the Remez exchange algorithm proposed below, it is inefficient both in terms of required storage and computation time.

III. Reformulation Of The Problem

In order to derive an efficient design procedure based on the Remez exchange algorithm, we first rewrite the problem (7)–(11), in such a way that paraunitariness (1) *and* flatness constraints (6) are built in the parameterization of $P(z)$. Making the change of variables $x = \cos\omega$ gives $P(e^{j\omega})$ as a $(L-1)$th order polynomial in x,

$$P(e^{j\omega}) = P(x) \tag{12}$$

(The same letter is used for different polynomials $P(z = e^{j\omega})$ and $P(x = \cos\omega)$, but the ambiguity should be easily resolved from the context.) Now paraunitariness (1) and flatness (6) constraints respectively become a symmetry and factorization condition of the form

$$P(x) + P(-x) = 2 \tag{13}$$

$$P(x) = (1+x)^K Q(x). \tag{14}$$

To find the general expression for $P(x)$ satisfying (13) and (14), we now solve the equation

$$(1+x)^K Q(x) + (1-x)^K Q(-x) = 2. \tag{15}$$

for $Q(x)$. This is classically done as follows.

We first find the general form of the difference between two distinct solutions to (15). Such a difference, $\Delta Q(x)$, satisfies

$$(1+x)^K \Delta Q(x) + (1-x)^K \Delta Q(-x) = 0. \tag{16}$$

Hence, $(1-x)^K$ divides $\Delta Q(x)$, $\Delta Q(x) = (1-x)^K S(x)$, and $S(x)$ is an odd polynomial, $S(x) = xR(x^2)$. The general form for $\Delta Q(x)$ is therefore $\Delta Q(x) = x(1-x)^K R(x^2)$.

Next, we can exhibit a particular solution to (15) of minimal degree: Since $\Delta Q(x)$ is a polynomial of degree at least $K+1$, there exists at most one solution to (15) of degree $\leq K$, corresponding to a solution $P(x)$ of minimal degree that we denote by $P_K(x)$. Finally, any solution to (15) is the sum of the particular solution to (15) just mentioned and a solution $\Delta Q(x)$. Returning to the initial problem, a characterization of (13), (14) is therefore

$$P(x) = P_K(x) + x(1-x^2)^K R(x^2). \tag{17}$$

For $K = 0$, $P_K(x) \equiv 1$. If we set $K = L/2$, we get $R(x) \equiv 0$ and $P_K(x)$ is identified as the maximally flat (Daubechies) solution of degree $2K-1$.

Now, $R(x^2)$ in (17) is a polynomial in x^2 of degree $L/2 - K - 1$ containing the $L/2 - K$ independent variables. Therefore, making the change of variables $y = x^2 = \cos^2\omega$ for $\omega \in [0, \omega_p]$, we can rewrite the initial problem (7)–(11) in terms of $R(y)$. Define the weighting function as

$$W(y) = \sqrt{y}(1-y)^K, \tag{18}$$

and the desired function as

$$D(y) = \frac{2 - \delta - P_K(\sqrt{y})}{W(y)}. \tag{19}$$

The optimization problem takes the form of an *unconstrained* minimization,

$$\min_{R(y)} \max_{y\in I} |E(y)| \tag{20}$$

where $I = [\cos^2\omega_p; 1]$ corresponds to the pass band interval and

$$\begin{aligned} E(y) &= W(y)(D(y) - R(y)), \\ &= 2 - \delta - P(x) \end{aligned} \tag{21}$$

is the weighted error.

This is almost in the form of a classical Chebyshev error minimization [7], but not quite: There is a slight irritation in that $D(y)$, hence, $E(y)$, depends on the tolerance δ. This creates an endless dependence loop since δ should be also equal to $\max_y |E(y)|$.

This problem is, in fact, brought by the simultaneous constraints of positivity (3) and flatness (6). When there is no flatness constraint ($K = 0$), Mintzer [6] and Smith and Barnwell [12] solve this problem in two steps: First, compute the frequency response using a classical equiripple design algorithm, in which δ in the desired response is removed. Then, raise the result by δ and rescale it to obtain the correct nonnegative frequency response. For $K > 0$, this approach becomes impossible since raising a frequency response by δ necessarily violates the flatness constraint (see Fig. 3).

It follows from the above discussion that classical design algorithms such as the Parks-McClellan algorithm [8] and the algorithm of Hoffstetter *et. al.* [4], in their classical version, can no longer be used for our problem. Therefore, in the remainder of this paper, we re-derive a modified Remez exchange algorithm adapted to formulation (18)–(21).

IV. An Alternation Theorem

A. Minimum Number of Alternations

The Remez exchange algorithm is based on the alternation theorem [7]. In this section, we show that the alternation theorem still applies to our problem (18)–(21). Instead of the usual trigonometric formulation [8], we use the more general polynomial formulation given in ([7], p. 468).

Alternation Theorem: Let I denote a closed set of disjoint intervals, $R(y)$ an rth order polynomial, $D(y)$ a given desired function, $W(y)$ a given positive function on I, and assume that both $D(y)$ and $W(y)$ are continuous on I. A necessary and sufficient condition that $R(y)$ is the unique rth order polynomial solution to (20) is that the weighted error, $E(y)$ (21), exhibit at least $(r+2)$ alternations,

$$E(y_i) = -E(y_{i+1}) = \pm \max_{y\in I} |E(y)| \tag{22}$$

for $y_1 < y_2 < \cdots < y_{r+2}$ in I.

In this theorem we set $W(y)$, $D(y)$ as given by (18), (19), and $I = [cos^2\omega_p, 1-\varepsilon]$, where ε is an arbitrarily small positive number, which is there to avoid the discontinuity of $D(y)$ at $y = 1$. The alternation theorem applies for any *fixed* value of δ in (19) and concludes that $E(y) = 2 - \delta - P(\cos\omega)$ exhibits at least $r + 2 = L/2 - K + 1$ alternations for $\omega \in [\varepsilon', \omega_p]$, where $\varepsilon' = \arccos\sqrt{1-\varepsilon} > 0$ is arbitrarily small.

In particular, let δ in (19) be the minimum tolerance $\delta^*$of the optimum solution to (7)–(11) (we have seen in Section II-A that this solution exists). Then, this solution is also the optimum solution to (20), and is therefore unique. Note that for $\delta = \delta^*$, we necessarily have $\min\max_y |E(y)| = \delta^*$. The alternation theorem is therefore valid for the optimum solution to our problem.

B. Maximum Number of Alternations

The maximum number of possible alternations can be estimated by numbering frequencies at which the slope of $E(y)$

vanishes in the pass-band. The number of such frequencies is equal to the number of roots of $P'(x)$, the derivative of $P(x)$. From (13), $P'(x)$ is an even polynomial in x of degree $L-2$, which has at most $L/2-1$ positive and $L/2-1$ negative roots. Hence, at most $L/2-1$ extremal frequencies occur in the pass band.

For $K=0$, we must add two possible alternations at the edges $\omega=0$and ω_p, which gives at most $L/2+1$ alternations. From the alternation theorem the number of alternations is exactly $L/2+1$, which includes alternations at the edges.

For $K>0$, (14) implies that among the possible $L/2-1$ negative roots of $P'(x)$, $K-1$ are located at -1, and hence, $K-1$ are located at 1. This leaves at most $L/2-1-(K-1)+1=L/2-K+1$ extremal frequencies in the pass band, including $\omega=0$, to which we add one possible alternation at ω_p. This gives a maximum of $L/2-K+2$ alternations. Since in this case the maximum number of alternations is equal to the minimum plus one, we can use classical arguments ([7], p. 4738) to show that an alternation occurs at ω_p (otherwise, two alternations would be removed). Now, at $\omega=0$, the alternation is negative (the ripple stands above the ideal response), while at $\omega=\omega_p$, the alternation is positive. Since the two alternations at the edges have different signs and since alternations must *alternate*, the total number of alternations must be even. This fixes the exact number of alternations to $L/2-K+1$ or $L/2-K+2$, whichever is even.

These results are summarized in the following Theorem.

A characterization of the unique solution to the problem (7)–(11) is that the pass-band error (21), for $\omega \in [0,\omega_p]$, exhibits

- *exactly $L/2+1$ alternations if $K=0$,*
- *exactly $L/2-K+1$ or $L/2-K+2$ alternations, whichever is even, if $K>0$.*

In all cases, $L/2-K+1$ alternations are a necessary and sufficient condition for optimality. Moreover, alternations always occur at $\omega=0$ and $\omega=\omega_p$.

V. Characteristics Of Optimal Wavelet Filters

A. Magnitude

The magnitude shape of best frequency selective wavelet filters is given by the theorem derived in the preceding section. A remarkable consequence of this theorem is that we can restrict the solutions to even values of $L/2-K$: This is because when $L/2-K$ is odd, the optimal solution for $K+1$ flatness constraints has the same number of alternations as the optimal solution for K flatness constraints. Hence, by application of our theorem, these two optimal solutions are equal. In other words, when $L/2-K$ is odd, the optimal solution $H(z)$ having K zeroes at $z=-1$ automatically has one more zero at $z=-1$.

It is interesting to note that, as another consequence of the theorem of the previous section, the shape of the optimal magnitude response depends only on $L/2-K$. The only exceptions are responses which do not vanish at half the sampling frequency, in other words, the "non-wavelet" filters ($L/2$ even and $K=0$). Fig. 4 illustrates that for optimum "wavelet filters," $L/2-K$ gives the (even) number of ripples

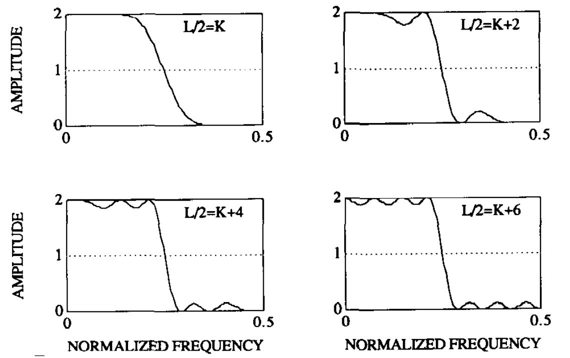

Fig. 4. Typical magnitude shapes of optimum solutions for several values of $L/2-K$ ($L=14$) and same normalized transition bandwidth = 0.05.

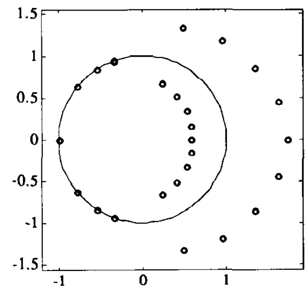

Fig. 5. A typical pole/zero distribution of the product filter $P(z)$ in the z-plane ($L=20$, $K=4$). The zero located at $z=-1$ has multiplicity 8, while the other zeroes located on the unit circle are double zeroes.

in the frequency response in the interval $(0,\pi)$, which are added to the Daubechies' maximally flat response. Half of this number gives the number of degrees of freedom used in the optimization problem addressed in this paper, since we have seen that the specifications are restricted to the first half-band.

B. Phase

So far we have described optimum solutions $P(z)$, which corresponds to the squared modulus of the transfer function of low-pass filter $H(z)$. In order to retrieve $H(z)$ itself, it is necessary to factor $P(z)$ into $H(z)$ and $H(z^{-1})$ according to (2). This is done by selecting zeroes of $H(z)$ (with their complex conjugates) from those of $P(z)$ [1], [12] (see Fig. 5). Each zero of $H(z)$ is chosen in a pair $(z_0, 1/z_0^*)$ of zeroes of $P(z)$, either inside or outside the unit circle, while zeroes of $P(z)$ located on the unit circle are retained with twice less multiplicity. Note that this root selection procedure will provide the phase information of $H(z)$, and there are several possible choices.

Based on the theorem derived in Section IV, we can calculate the total number of choices for $H(z)$ from a given solution $P(z)$: That will depend on the number of zeroes of $P(z)$ which lie outside the unit circle. Now, the number of zeroes located on the unit circle is given by the number of alternations in the stop band at which the magnitude response vanishes. It is easily seen, using the theorem derived in the preceding section, that the total number of such zeroes, *counted with their multiplicity* at $\omega = \pi$, does not depend on K. This gives a total of L zeroes of $P(z)$ on the unit circle.

The remaining $L - 2$ zeroes determine the choice of $H(z)$. They occur either in quadruple configuration $(z_0, z_0^*, 1/z_0, 1/z_0^*)$ or in real pairs $(z_0, 1/z_0)$. We observed that there are at most one such real pair in all cases. Therefore, the possible selections are one for each quadruple and, if $L - 2$ is an odd multiple of two, one for the real pair. This gives a total of $2^{\lfloor L/4 \rfloor}$ different filter solutions $H(z)$ to (2), corresponding to different phases [1], [12]. Since half of the solutions are the time-reversals of the other half, the number of solutions within time-reversal is $2^{\lfloor L/4 \rfloor - 1}$. In the example of Fig. 5, this gives 16 choices.

VI. A Remez Exchange Algorithm

In this section, we derive our main filter design algorithm, based on the alternation theorem derived earlier in Section IV. This derivation and the various techniques used in the algorithm are similar to what has been done for classical Remez exchange algorithms [4], [8]. However, there are important differences, which are stressed throughout this section.

The alternation theorem derived in Section IV states that a necessary and sufficient condition for the product filter $P(z)$ to be optimum is that

$$W(y_k)(D(y_k) - R(y_k)) = (-1)^{k+1}\delta^* \tag{23}$$

where δ^* is the optimum error, $W(y)$, $D(y)$ and $R(y)$ were defined in Section IV, and y_k, $k = 1, \ldots, L/2 - K + 1$, are $L/2 - K + 1$ alternations in $[cos^2\omega_p, 1]$. Note that $y_1 = cos^2\omega_p$. The corresponding ripple lies below the ideal response, giving the sign $(-1)^{k+1}$ in (23). This condition serves as a basis for efficient algorithms for finding the optimal filter $P(z)$.

The theorem requires only $L/2 - K + 1$ alternations, while the optimal solution exhibits in fact $L/2 - K + 2$ alternations in the case $L/2 - K$ even. (We have seen in Section II-A that we could always restrict ourselves to this case.) Therefore, the situation is almost the same as in the classical design of extraripple filters (without the flatness constraints), except for different definitions of $W(y)$, $D(y)$ and $R(y)$. The main difference is that the optimum error δ^* appears in the definition of $D(y)$ (19). We therefore rewrite (23) in terms of δ^* by letting

$$D'(y) = D(y) + \frac{\delta^*}{W(y)} = \frac{2 - P_K(\sqrt{y})}{W(y)}. \tag{24}$$

This gives the fundamental necessary and sufficient condition

$$R(y_k) + \frac{(1 + (-1)^{k+1})\delta^*}{W(y_k)} = D'(y_k), \tag{25}$$

from which efficient algorithms can be derived. Two classical design algorithms, similar to the Remez multiple exchange algorithm of approximation theory, can be mimicked from (25).

The Hofstetter-Oppenheim-Siegel algorithm [4] takes the extra ripple into account. Given a fixed value of δ, it begins by making an initial estimate of the y_k, then computes $R(y)$ using a Lagrangian interpolation formula and the band-pass error, determines the new set of y_k as the points at which the band-pass error is extremum and starts another iteration for this new set. After convergence, we obtain optimum filters of specified tolerance δ, but the transition bandwidth cannot be specified *a priori.*

In the remainder of this section, we describe the second type of algorithm in more detail. This algorithm is similar to the multiple exchange algorithm proposed by Parks and McClellan [8]. A MATLAB listing is provided in Appendix A.

A. Description of the Algorithm

The derivation of the algorithm itself can be straightforward, using condition (25). Formally, the only difference between a classical Remez exchange algorithm is that δ^* in (25) is modulated by $1 + (-1)^{k+1}$ instead of the usual $(-1)^k$. However, this remark hides the important conceptual problem which we have already mentioned, namely, the dependence of the desired response on the optimum error δ^*. The following description circumvents this problem by working with successive values of δ: At each iteration, not only the values of the estimated error δ, but also the optimization problem, *via* the desired response, change. These successive optimization problems ultimately converge to the original optimization problem associated to δ^*, as the estimated error δ tends to δ^*.

Using (25) as a starting point, we now describe the step-by-step algorithm following the description made in [7]: First, an initial estimate of the y_k's, $k = 1, \ldots, L/2 - K + 1$, is made. Then, (25), considered as a linear system of $L/2 - K + 1$ equations in the $L/2 - K + 1$ unknowns r_n and δ, is used to compute δ by the formula

$$\delta = \frac{\sum_k L_k^{-1} D'(y_k)}{\sum_k L_k^{-1}(1 + (-1)^{k+1})/W(y_k)}. \tag{26}$$

where $L_k = \prod_{i \neq k}(y_k - y_i)$. This equation is classically obtained as in the Parks/McClellan algorithm, by exploiting the structure of the underlying Vandermonde matrix of system (25). Then $R(y)$, where y belongs to a fine grid in the interval $[cos^2\omega_p, 1]$, is computed from its values at the y_k's, $k = 1, \ldots, L/2 - K$, which from (25) are given by

$$R(y_k) = D'(y_k) - (1 + (-1)^{k+1})\delta/W(y_k). \tag{27}$$

The following Lagrangian interpolation formula (in barycentric form) is used.

$$R(y) = \frac{\sum_k R(y_k) L_k'/(y - y_k)}{\sum_k L_k'/(y - y_k)} \tag{28}$$

where $L_k' = (y_k - y_{L/2-K+1})/L_k$. Now, $R(y)$ is available on the desired interval without the need to solve (25) for the coefficients of $R(y)$. It is used to evaluate the band-pass error, $E(y) = 2 - \delta - P_K(\sqrt{y}) - W(y)R(y)$. The new set of extremal frequencies y_k, $k = 2, \ldots, L/2 - K + 1$, is then determined as the locations at which the slope of $E(y)$ vanishes. Note that from the discussion of Section IV, there can be no more than $L/2 - K$ extrema in the interval considered (excluding the extra ripple at $y = 1$ if $K > 0$). From here, another iteration starts.

To compute $P_K(\sqrt{y})$ in (24), where $\sqrt{y} = x$, we note that from (14), $(1+x)^{K-1}$ divides $P'_K(x)$, the derivative of $P_K(x)$. From (13), $P'_K(x)$ is an odd polynomial of degree $2K - 2$. It follows that $P'_K(x) \propto (1 - x^2)^{K-1}$. Integrating this relation yields $P_K(x) = 1 + xD_K(x^2)$, where $D_K(x) = \sum_n d_n x^n$ is given by

$$d_n \propto \frac{(-1)^n}{2n+1}\binom{K-1}{n}. \quad (29)$$

(The symbol $\binom{K-1}{n}$ denotes binomial coefficients.) The d_n's should be normalized such that their sum is 1. Note that $D_K(y) \equiv 0$ if $K = 0$.

If the current extremal frequencies did not change from the ones at the previous iteration, the algorithm has converged.The impulse response of $P(z)$ is then determined from $R(y)$ using formulae based on the inverse DFT, which are given in the next section.

B. Computing the Optimal Impulse Response

The last step in the algorithm computes the impulse reponse of $P(z)$from the values of $R(y)$ on a set of points in the interval I. To do this, we compute the transfer function $R(z)$, corresponding to $R(y = \cos^2\omega)$, from values of $R(y)$ on regularly spaced frequencies $\omega_i = i\pi/N$, $i = 0, \ldots, N-1$, where $N = L - 2K - 1$. It is easily seen that the coefficients of $R(z)$ are given by an inverse DFT formula which reduces to

$$r_n = \frac{1}{N}\sum_{i=0}^{N-1}(-1)^i R(\cos^2\frac{i\pi}{N})cos(2n+1)\frac{i\pi}{N}. \quad (30)$$

This formula is preferred over an FFT algorithm for short to medium lengths. Finally, to recover $P(z)$, we need to add the maximally flat solution $P_K(z)$ of length $4K - 1$, which itself can be computed using a closed-form expression, given below[1].

From (6), $(1 + z^{-1})^{2K-1}$ divides both $P_K(z)$ and its derivative with respect to z^{-1}, $P'_K(z)$. Therefore, it also divides $(2K-1)P_K(z) - z^{-1}P'_K(z)$. From (1), $P_K(z)$ is of the form $z^{-(L-1)} + A(z^2)$ and the above expression is therefore a polynomial in z^{-2}, hence, a multiple of $(1 - z^{-2})^{2K-1}$. By equality of degrees, $(1 - z^{-2})^{2K-1}$ and $(2K - 1)P_K(z) - z^{-1}P'_K(z)$ are equal up to a multiplicative constant. We obtain that the coefficients of $P_K(z)$, p_n^K, are given by $p_{2K-1}^K = 1$, $p_{2n-1}^K = 0$ for $n \neq K$, and

$$p_{2n}^K \propto \frac{(-1)^n}{2K-1-2n}\binom{2K-1}{n} \quad (31)$$

for $n = 0, \ldots, L-1$. The p_{2n}^K's should further be normalized such that $\sum_n p_{2n}^K = P_K(1) - 1 = 1$.

C. Computing the Roots

As shown in [2], [12] and Section V-B, the phase of paraunitary solutions $H(z)$ corresponding to a given $P(z)$ are obtained from the roots of $P(z)$. However, a root extraction routine is very sensitive to multiple roots at $z = -1$ in $P(z)$, so computing the roots of $P(z)$ is better done from a factorized form of $P(z) = (1 + z^{-1})^{2K}Q(z)$. This factorization is readily obtained from (17) and (23). However, it requires the precalculation of the corresponding quotient polynomial $Q_K(z)$ for the maximally flat solution $P_K(z)$, which can be obtained as follows.

The equality

$$(2K-1)P_K(z) - z^{-1}P'_K(z) = c(1 - z^{-2})^{2K-1},$$

proved in the preceding section, can be rewritten in terms of $Q_K(z)$ as

$$(2K-1-z^{-1})Q_K(z) - (z^{-1}+z^{-2})Q'_K(z) = c(1-z^{-1})^{2K-1}.$$

The coefficients of $Q_K(z)$ can be easily determined by induction as

$$q_n^K \propto \binom{2K-2}{n}^{-1}\sum_{i=0}^{n}(-1)^i\binom{2K-1}{i}^2 \quad (32)$$

for $n = 0\ldots, 2K - 2$. They should be normalized such that $\sum_n q_n^K = 2^{-2K}$.

D. Flowchart of the algorithm

Let us summarize the description of our algorithm. The various steps are identified in the MATLAB listing provided in Appendix A.

Preliminaries

1) Step 1: If $L/2 - K$ is odd, replace K by $K + 1$ (see remark in Section V-A) and set $N = L/2 - K$.
2) Step 2: Compute Daubechies response of order K using (31).
3) Step 2′: If required, compute Daubechies factor $Q_K(z)$ using (32).
4) Step 3: If $K = L/2$, output Daubechies solution and stop.
5) Step 4: Compute $D_K(y)$, which is necessary for the computation of $P_K(\sqrt{y})$, using (29).

Remez exchange algorithm proper

1) Step I: Initial estimate of the $N+1$ extremal frequencies y_k (see the remark at the beginning of Section VII)
2) Step II: Compute δ using (26).
3) Step III: Compute $R(y)$ on a fine grid using Lagrangian interpolation formula (28).
4) Step IV: Compute new extremal frequencies y_k from $E(y)$. If the extremal frequencies did not change, go to Step A. Otherwise go to Step II.

Output solution

1) Step A: Compute impulse response using (23) and the result of Step 2 as seen above.
2) Step A′: If required, compute the roots using the result of Step 2′ as seen above.

[1] Although closed-form expressions for maximally flat filters are available [3], [5], we re-derive special forms adapted to our algorithm in Sections VI-B and VI-C. Some of them should be known, although we were unable to find them in the literature.

VII. Results And Discussion

Thanks to the alternation theorem, the algorithm presented in the preceding section necessarily provides the optimal solution to our problem, provided that it converges. In fact, because this algorithm is similar to a classical Remez exchange algorithm, except for the small change in (25), where $(-1)^{k+1}$ is replaced by $(1 + (-1)^{k+1})$, it can be shown that the algorithm does converge to the solution of the initial problem described in Section II-A. A rigorous proof of this is available from the authors [10].

Therefore, the initial estimate for the extremal frequencies does not affect convergence, but rather the number of iterations needed prior to convergence. Appendix A gives a formula for the initial set of y_k's, for which the number of iterations is reduced by one or two compared to the case where the initial frequencies are regularly spaced. With this initial set, the number of iterations seldom exceeds 3 for lengths $L \leq 20$, which correspond to a product filter of order ≤ 40. Compared to linear programming techniques, our method is therefore much less time consuming and requires less memory occupation. However, the accuracy of optimum solutions were found to be reasonably good for both methods (less than 10^{-5} difference on filter taps for short to medium lengths, which correspond to most practical cases).

In addition, the number of independent variables in our algorithm, $L/2 - K$, was shown in Section IIII to be minimal for our problem. Note that for $K = 0$, the resulting algorithm is even more efficient than a general Parks-McClellan algorithm used to design the half-band product filter of length $2L-1$—as suggested in [6], [12]—since the number of variables has been reduced by a factor 2. For a fixed length L, the number of variables decreases as K is taken larger, and the algorithm is faster. For $K = L/2$, the maximally flat solution is readily obtained from (31) or (32). Fig. 6 illustrates the behavior of the optimal solution of length L compared to the "ideal response," a maximally flat filter meeting K flatness constraints.

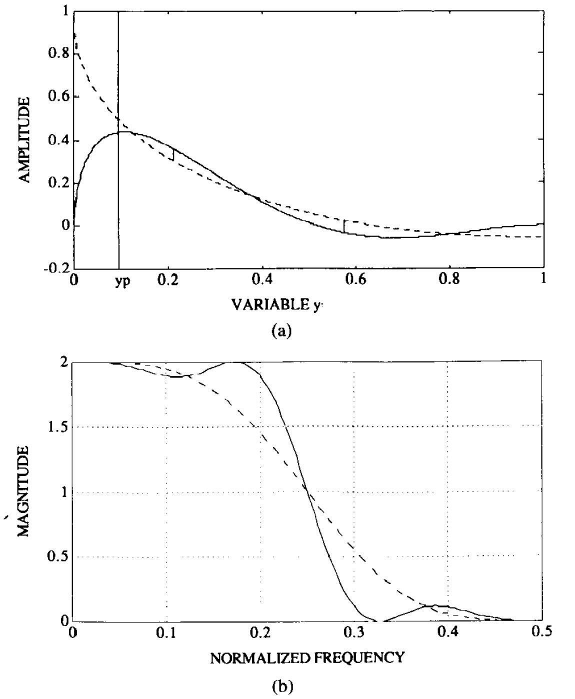

Fig. 6. (a). Weighted desired function $W(y)D(y)$ and equiripple optimum weighted polynomial $W(y)R(y)$, for $y = \cos^2 \omega \in [0, 1]$, $L = 8$, $K = 2$, and normalized transition bandwidth 0.1. The cut-off frequency corresponds to the value y_p. The alternations in the variable y are indicated by segments. (b). Optimum frequency response ($L = 8$, $K = 2$) and corresponding desired maximally flat response ($L = 4$, $K = 2$). The additional degrees of freedom are used to provide an equiripple solution.

A. Regularity versus Stop-Band Attenuation

Using this algorithm, we obtained a number of filters with balanced stop-band attenuation and regularity (see Fig. 7). Regularity was measured, using the optimal estimation algorithm given in [9], as the Sobolev regularity order of the corresponding wavelets. A regularity order greater than N means that the wavelet is N times continuously differentiable. The measure used here gives the minimum regularity order that can be attained for all solutions $H(z)$ (of different phases) corresponding to a given product filter $P(z)$ Adding 1/2 to this measure gives the maximum possible regularity order. The results are shown in Fig. 7. An alternative would have been to use the "exact" Hölder definition of regularity for all possible wavelets determined from $P(z)$ [9]. We would, however, obtain the same behavior (within a small shift in the regularity axis of Fig. 7).

Fig. 7 shows that regularity of Mintzer-Smith-Barnwell filters [6], [12] is greatly improved by imposing a few flatness constraints, while preserving frequency selectivity. Conversely, selectivity of Daubechies filters can be greatly improved by relaxing a few flatness constraints, which results in a small loss of regularity for small filters. For larger filters ($L \geq 20$), both frequency selectivity and regularity can be improved this way: this is another counter example [2] to the general belief that, for fixed L, regularity is an increasing function of K.

B. Phase and Group Delay Deviation

It is well known [12] that the phase of paraunitary solutions $H(z)$ cannot be chosen linear for $L \leq 4$. However, if zeroes of product filter $P(z)$ are selected in a suitable manner, then $H(z)$ will be approximately linear phase [1], [12]. To illustrate this effect, we use the global variation of the group delay in the pass-band as a measure, in samples, of the distance to linear phase. Denoting the group delay of $H(z)$ by $g(\omega)$, this measure is

$$\max_{\omega \in [0, \omega_p]} g(\omega) - \min_{\omega \in [0, \omega_p]} g(\omega). \tag{33}$$

Had $H(z)$ be linear phase, this expression would vanish. Solutions that are "closest to linear phase" minimize (33) for the same magnitude response.

Fig. 8 shows that solutions closest to linear phase have reasonably small group delay variation (about 0.2 to 2.5 samples for $L \leq 20$). Solutions closest to linear phase are

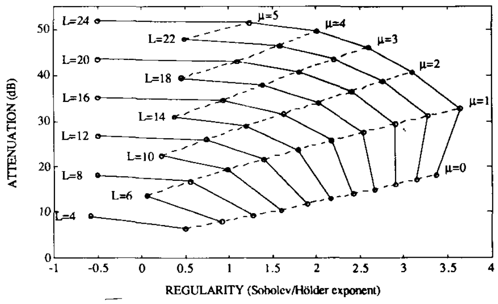

Fig. 7. Attenuation (in dB) versus regularity orders for the families of optimum filters obtained with normalized transition bandwidth set to 0.14. The regularity orders shown in this plot are optimal measures of regularity [9] for a given magnitude response $P(e^{j\omega})$. Solid lines correspond to filters $H(z)$ of constant length L and different degrees of flatness $0 \leq K \leq L/2$. Dotted lines correspond to fixed values of $L/2 - K$, the number of ripples in the magnitude response of $H(z)$. Daubechies filters [1] correspond to $L/2 - K = 0$, while Mintzer-Smith-Barnwell filters [6], [12] correspond to the least values of regularity, which are negative for $K = 0$.

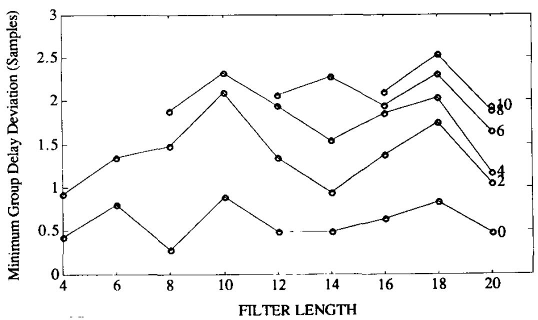

Fig. 8. Global group delay variation in the pass-band of solutions $H(z)$that are closest to linear phase, versus filter length L for several values of $L/2 - K$, which correspond to the number of ripples in the frequency response, as seen in Fig. 4. Solutions closest to linear phase are obtained for $L/2 - K = 0$, which corresponds to Daubechies filters [2].

obtained for Daubechies filters ($K = L/2$). We observed that for any value of K, solutions closest to linear phase are not necessarily obtained by selecting zeroes of $P(z)$ alternatively from inside or outside the unit circle, as was suggested for the case $K = 0$ in [12]. Also, solutions farthest from linear phase, for which the group delay variation is largest, are not always the minimum or maximum phase solutions.

VIII. Conclusion

This paper has provided an efficient design procedure, based on the Remez exchange algorithm, which computes optimum wavelet filters, i.e., best frequency selective paraunitary filter bank solutions, for a given transition bandwidth and flatness constraints at the Nyquist frequency. These optimum solutions satisfy the alternation theorem and, when not maximally flat, exhibit equiripple frequency responses. This procedure is much more efficient than procedures based on linear programming, both in terms of memory and computation time. Using this procedure, we obtain a large number of "orthonormal wavelet filters" [1], with balanced regularity, frequency selectivity, number of taps, and phase. We have shown that compared to known "wavelet" filters [1], [6], [12], which are obtained as special cases, regularity and/or frequency selectivity can be greatly improved by imposing or relaxing flatness constraints.

IX. Appendix
Matlab Function Implementing The Algorithm

```
function [p,r]=remezwav(L,K,B)
% P=REMEZWAV(L,K,B) gives impulse res-
% ponse of maximally frequency selective
% P(z), product filter of paraunitary
% filter bank solution H(z) of length L
% satisfying K flatness constraints
% (wavelet filter), with normalized
% transition bandwidth B (optional
% argument if K==L/2).
% [P,R]=REMEZWAV(L,K,B) also gives the
% roots of P(z), which can be used to
% determine H(z).

% Author: Olivier Rioul, Nov. 1, 1992.
% For MATLAB 4.0 or 4.1

computeroots=(nargout>1);

%%%%%%% STEP 1 %%%%%%%%%
if rem(L,2),error('L must be even');end
if rem(L/2-K,2), K=K+1, end
N=L/2-K;
%%%%%%% STEP 2 %%%%%%%%
% Daubechies solution
% PK(z)=z^(-(2K-1))+AK(Z^2)
if K==0, AK=0;
else
 binom=pascal(2*K,1);
 AK=binom(2*K,1:K)./(2*K-1:-2:1);
 AK=[AK AK(K:-1:1)];
 AK=AK/sum(AK);
end
%%%%%%% STEP 2' %%%%%%%%
% Daubechies factor
% PK(z)=((1+z^(-1))/2)^(2K) QK(z)
if computeroots & K>0
 QK=binom(2*K,1:K);
 QK=QK.*abs(QK);
 QK=cumsum(QK);
 QK=QK./abs(binom(2*K-1,1:K));
 QK=[QK QK(K-1:-1:1)];
 QK=QK/sum(QK)*2;
end
%%%%%%% STEP 3 %%%%%%%%
% output Daubechies solution PK(z)
if K==L/2
 p=zeros(1,2*L-1);
 p(1:2:2*L-1)=AK; p(L)=1;
```

```
 if computeroots
  r=[roots(QK); -ones(L,1)];
 end
 return
end
%%%%%%%% STEP 4 %%%%%%%%
% Daubechies polynomial
% PK(x)=1+x*DK(x^2)
if K==0, DK=0;
else
 binom=pascal(K,1);
 binom=binom(K,:);
 DK=binom./(1:2:2*K-1);
 DK=fliplr(DK)/sum(DK);
end

wp=(1/2-B)*pi; % cut-off frequency
gridens=16*(N+1); % grid density
found=0; % boolean for Remez loop

%%%%%%%% STEP I %%%%%%%%
% Initial estimate of yk
a=min(4,K)/10;
yk=linspace(0,1-a,N+1);
yk=(yk.^2).*(3+a-(2+a)*yk);
yk=1-(1-yk)*(1-cos(wp)^2);
ykold=yk;

iter=0;
while 1 % REMEZ LOOP
iter=iter+1;

%%%%%%%% STEP II %%%%%%%%
Compute delta Wyk=sqrt(yk).*((1-yk).^K);
Dyk=(1-sqrt(yk).*polyval(DK,yk))./Wyk;
for k=1:N+1
dy=yk-yk(k); dy(k)=[];
dy=dy(1:N/2).*dy(N:-1:N/2+1);
Lk(k)=prod(dy);
end
invW(1:2:N+1)=2 ./Wyk(1:2:N+1);
delta=sum(Dyk./Lk)/sum(invW./Lk);
%%%%%%%% STEP III %%%%%%%%
compute R(y) on fine grid
Ryk=Dyk-delta.*invW; Ryk(N+1)=[];
Lk=(yk(1:N)-yk(N+1))./Lk(1:N);
y=linspace(cos(wp)^2,1-K*1e-7,gridens);
yy=ones(N,1)*y-yk(1:N)'*ones(1,gridens);
% yy contains y-yk on each line
ind=find(yy==0); % avoid division by 0
if ~isempty(ind)
yy(ind)=1e-30*ones(size(ind));
end
yy=1./yy;
Ry=((Ryk.*Lk)*yy)./(Lk*yy);
%%%%%%%% STEP IV %%%%%%%%
Find next yk
Ey=1-delta-sqrt(y).* ...
(polyval(DK,y)+((1-y).^K).*Ry);
k=find(abs(diff(sign(diff(Ey))))==2)+1;
% N extrema
if length(k)>N
% may happen if L and K are large k=k(1:N);
end
yk=[yk(1) y(k)];
% N+1 extrema including wp
if K==0, yk=[yk 1]; end
% extrema at y==1 added
if all(yk==ykold), break; end
ykold=yk;

end % REMEZ LOOP
fprintf('%g iterations\n',iter);

%%%%%%%% STEP A %%%%%%%%
% Compute impulse response
w=(0:2*N-2)*pi/(2*N-1);
y=cos(w).^2;
yy=ones(N,1)*y-yk(1:N)'*ones(1,2*N-1);
ind=find(yy==0);
if ~isempty(ind)
 yy(ind)=1e-30*ones(size(ind));
end
yy=1./yy;
Ry=((Ryk.*Lk)*yy)./(Lk*yy);
Ry(2:2:2*N-2)= -Ry(2:2:2*N-2);
r=Ry*cos(w'*(2*(0:N-1)+1));
% partial real IDFT done
r=r/(2*N-1);
r=[r r(N-1:-1:1)];
p1=[r 0]+[0 r];
pp=p1; % save p1 for later use
for k=1:2*K
 p1=[p1 0]-[0 p1];
end
if rem(K,2), p1=-p1; end
p1=p1/2^(2*K+1);
p1(N+1:N+2*K)=p1(N+1:N+2*K)+AK;
% add Daubechies response:
p(1:2:2*L-1)=p1; p(L)=1;
%%%%%%%% STEP A' %%%%%%%%
% Compute roots
if computeroots
 Q(1:2:2*length(pp)-1)=pp;
 for k=1:2*K
  Q=[Q 0]-[0 Q];
 end
 if rem(K,2), Q=-Q; end
 Q=Q/2;
 if K>0 % add Daubechies factor QK
  Q(2*N+1:L-1)=Q(2*N+1:L-1)+QK;
 else
  Q(L)=1;
 end
 r=[roots(Q); -ones(2*K,1)];
end
```

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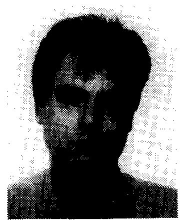

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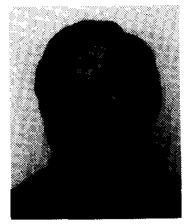

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