REGULAR WAVELETS: THEORY AND ALGORITHMS

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Abstract. Regular wavelets are generated by binary subdivision schemes. In this paper, we use a polynomial description of such schemes to study the existence and Hölder regularity of wavelets. Sharp regularity estimates are derived. They are optimal—except for pathological cases—and can be implemented easily on a computer.

Introduction

Orthonormal [3] or biorthogonal [2] wavelets can be seen as limit functions of "binary subdivision schemes" [5]. Such schemes also find application in geometric design [5] and image coding using filter banks [8,9]. Whether limit functions are regular or not may be relevant in these applications, and this topic is of growing interest since the discovery of compactly supported wavelets by Daubechies [3].

We start by describing subdivision schemes. Throughout this paper we consider real-valued discrete sequences u_n ($n \in \mathbf{Z}$) of finite length. Given the initial sequence $\delta_n = 1$ if $n = 0$, 0 otherwise, and a "subdivision mask [5]" g_n, $n = 0, \ldots L-1$, a "binary subdivision scheme" is the collection of sequences $g_n^j = \mathcal{G}^j\{\delta_n\}$, defined by repeated application of the operator

$$u_n \xrightarrow{\mathcal{G}} v_n = \sum_k u_k \, g_{n-2k}. \tag{1}$$

The extension to other initial sequences is trivial [7]. In polynomial notation, we have

$$U(X) \xrightarrow{\mathcal{G}} V(X) = G(X)\,U(X^2), \tag{2}$$

where $U(X) = \sum_n u_n X^n$ is associated to any causal sequence u_n (Laurent polynomials can be used for non-causal sequences). Iterating (2) gives

$$G^j(X) = G(X)\,G(X^2)\,G(X^4)\ldots G(X^{2^{j-1}}). \tag{3}$$

as the polynomial of degree $(2^j-1)(L-1)$ associated to the sequence g_n^j. Now, the graphs of g_n^j, when plotted against $n2^{-j}$, may converge—in a sense to be defined later—to a limit function $\phi(x)$, called the "scaling function" [2,3]. The construction of wavelet $\psi(x)$ is now in one quick step:

$$\psi(x) = \sum_k h_k \, \phi(2x-k). \tag{4}$$

where h_n is another set of coefficients [2,3]. Since mask g_n is of finite length, all functions considered here have compact support included in the interval $(0, L-1)$. Therefore, $\phi(x)$ itself can be written as linear combinations of $\psi(x/2-k)$; the convergence and regularity properties of $\phi(x)$ and $\psi(x)$ are the same, and we can restrict ourselves to the study of $\phi(x)$.

Our aim is to find necessary and sufficient conditions on $G(X)$ such that 1) convergence of the g_n^j's holds to a limit function $\phi(x)$, and 2) $\phi(x)$ is regular, i.e., continuous, possibly with several continuous derivatives. $\phi(x)$ is rarely obtained as an explicit function of x, and the applications only require g_n^j. Therefore, regularity, defined on functions of a continuous variable, would be understood better if it were expressed in terms of these discrete sequences. In the following, we adopt this "discrete" approach, which is found to be powerful: It leads to regularity estimates that are, in contrast with earlier ones [1,3,4], easily implementable, optimal, and of general applicability. A complete mathematical treatment can be found in [7].

1 The Stability Condition

Owing to the limit process, all "discrete" regularity conditions presented in this paper will imply the corresponding regularity properties of $\phi(x)$ [7], and always provide a lower bound for regularity. To obtain also an upper bound, i.e., to get optimal results, we need to have the converse implications. This can be done using the formula

$$\Phi^j(X) = \Phi(X)G^j(X) \tag{5}$$

where $\Phi^j(X)$ is the polynomial associated to the sequence $\phi(n2^{-j})$ and $\Phi(X) = \Phi^1(X)$. This relation is easily derived from the "two-scale difference equation" [4] satisfied by $\phi(x)$, and gives a simple method for computing exact values of $\phi(x)$ at dyadic rationals, by relating them to g_n^j [7]. Now, to obtain regularity conditions on g_n^j from those on $\phi(x)$, the inverse polynomial $1/\Phi(X)$ in (5) must be numerically stable for finite-length sequences. This gives the following

Definition 1.1 ([7]) *A binary subdivision scheme converging to a limit function $\phi(x) \not\equiv 0$ is "stable" if there exists $x \in \mathbf{R}$ such that*

$$\sum_n \phi(n+x)\, e^{in\omega} \neq 0 \quad \textit{for all } \omega \in \mathbf{R}. \tag{6}$$

(A precise definition of convergence is given in the next section.) This slightly restricts the choice of the scaling sequence g_n: There is an exceptional class of limit functions $\phi(x)$ for which the regularity estimates derived in this paper will not always be optimal. For example, any polynomial mask $G(X)$ divisible by $(X^2 - e^{i\omega})$, $\omega \neq 0$, yields unstability, and I conjecture that the converse implication holds [7]. For orthonormal wavelets [3], this unstability condition is never satisfied [8]. In the general case of biothogonal wavelets [2], it is theoretically possible, but rare in practice [8], to get unstability. Even when the unstability condition is met, a trick shown in section 4 solves the problem.

2 Uniform Convergence and C^N Wavelets

Several definitions of convergence of the graphs of $\{g_n^j\}$—plotted against $n2^{-j}$—to $\phi(x)$ were proposed [3,5]. A popular approach [3] is to define convergence for staircase functions whose values at $x = n2^{-j}$ are the g_n^j's. Another flexible definition for uniform convergence [7] is

$$\lim_{j\to\infty} \sup_x |\phi(x) - g_{n_j}^j| = 0 \tag{7}$$

The flexibility comes from the arbitrary choice of integers n_j satisfying

$$|n_j - x2^j| \leq \text{Const.} \tag{8}$$

The same definition, without the supremum, can be used for pointwise convergence [7].

Theorem 2.1 ([7]) *Assume that a binary subdivision scheme converges pointwise to a limit function $\phi(x)$ for all $x \in \mathbf{R}$. If the convergence is uniform, then $\phi(x)$ is continuous. The converse is true in the case of stability* (6). *Uniform convergence holds if and only if*

$$G(1) = 2, \quad G(-1) = 0, \text{ and} \tag{9}$$

$$\max_n |g^j_{n+1} - g^j_n| \to 0 \text{ as } j \to \infty. \tag{10}$$

Moreover, there exists $\alpha > 0$ such that

$$\max_n |g^j_{n+1} - g^j_n| \leq c\, 2^{-j\alpha}. \tag{11}$$

Therefore, for regular limit functions, the type of convergence of stable g^j_n is uniform, which justifies our choice of uniform convergence. Moreover, we have a necessary and sufficient condition. The two basic conditions (9) have been known for some time [2,3,5]. This first one is simply a normalization requirement. For orthonormal wavelets, $G(1) = \sqrt{2}$, convergence may hold with normalization factor $2^{j/2}$ on g^j_n. The second one ensures that the g^j_n's, for large j, do not rapidly oscillate in n [7], which turns out to be crucial for coding applications [8]. Finally, it can be shown [7] that all definitions of uniform convergence proposed in the literature [3,5] are equivalent, and this is implied by the unique characterization (10).

We thus have a general characterization of continuity. We shall see in section 3 that (11) in fact implies that $\phi(x)$ is Lipschitz of order α, which is stronger than continuity. However, even when $\phi(x)$ is required to be continuous, it may not appear to be smooth at all, as shown in Fig. 1. In fact, condition (11) requires that the "slopes"

$$\delta g^j_n = (g^j_{n+1} - g^j_n)/2^{-j} \tag{12}$$

of the discrete curve g^j_n, plotted against $n2^{-j}$, do not increase faster than $2^{j(1-\alpha)}$ as $j \to \infty$. But they can still increase indefinitely if $\alpha < 1$, leading to a "fractal-like" curve as in Fig. 1.

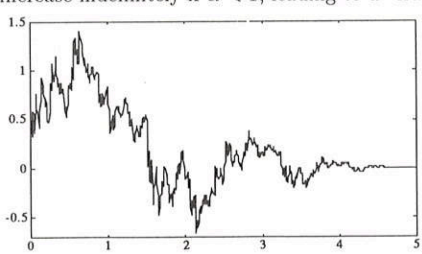

Figure 1: *An exemple of continuous scaling function [8]. Its (best) Sobolev regularity order is negative ($-0.071\ldots$), but its Hölder regularity order is $r \approx 0.2$.*

Therefore, to obtain smoother limit functions, we should require more than continuity. To characterize regularity order N ($\phi(x) \in C^N$) on g^j_n, we note that the role of the Nth order derivative of $\phi(x)$ is played in the discrete domain by the *Nth order finite difference* of g^j_n.

The first-order finite differences are simply the sequence of slopes (12). Applying N times the operator δ gives the Nth order finite difference $\delta^N g_n^j$.

First, $\delta^N g_n^j$ can be expressed as binary subdivision schemes, provided $G(X)$ has at least N roots at $X = -1$. The corresponding subdivision mask is [7] $G(X)(\frac{1+X}{2})^{-N}$. Then, we can apply the above results to $\delta^N g_n^j$. The graphical interpretation of this is the same as above, but applies to the sequence of slopes, or slopes of slopes, etc., leading to smoother and smoother evolutions of g_n^j.

Theorem 2.2 ([7]) *If the sequence of the Nth-order finite differences $\delta^N g_{n_j}^j$ (where n_j satisfies* (8)*) uniformly converges as $j \to \infty$, then $\phi(x)$ is C^N. The converse is true if $\phi(x)$ is stable. In addition, $\delta^k g_{n_j}^j$ converges uniformly to $\phi^{(k)}(x)$, the kth order derivative of $\phi(x)$, for $k = 0$, ..., N, and $G(X)$ has at least $N+1$ roots at $X = -1$.*

The existence of zeroes at $X = -1$ is a constructive result, already obtained in several papers [2, 5], which gives a simple rule for constructing regular wavelets. For example, Daubechies [3] constructed her orthonormal wavelets by imposing as many zeroes at $X = -1$ as possible in $G(X)$ for a given mask length. However, the effect of zeroes at $X = -1$ may be killed by the *other* zeroes of $G(X)$, whose effect is always more or less destructive for regularity [8]. In other words, Theorem 2.2 states that the regularity order of (stable) $\phi(x)$ is strictly bounded by the number of zeros at $X = -1$ in $G(X)$, but such zeroes not sufficient to obtain a given regularity order [7].

3 Estimating Hölder Regularity

To quantify regularity accurately, we now extend the definition of regularity order to real-valued numbers. There are several ways of doing this, the most common ones use *Sobolev* spaces H^s and *Hölder* spaces $\dot{C}^r$. Sobolev definition is a popular spectral approach to regularity [2,3,10]: $\phi(x)$ has Sobolev regularity order r if it belongs to the Sobolev space $H^{r+1/2}$. This can be easily tackled by simple estimations on $|G(e^{i\omega})|$ [3]. However, only the modulus of $G(e^{i\omega})$ is taken into account—phase information is ignored— and the best Sobolev regularity order may be negative, even though $\phi(x)$ is in fact continuous (see Fig. 1).

These drawbacks are avoided with the Hölder definition of regularity, which was introduced recently for wavelets [4]. Hölder (or Lipschitz) spaces $\dot{C}^\alpha$, $0 < \alpha \leq 1$, interpolate between C^0 and C^1; a $\dot{C}^\alpha$-function will be said to be regular of order α. For higher regularity orders $r = N + \alpha$, $N = 1, 2, \ldots$, and $0 < \alpha \leq 1$, the Lipschitz definition is used on the Nth derivative of $\phi(x)$. The difference between Hölder and Sobolev regularity only depends on the phase of $G(e^{i\omega})$, and is always less than 1/2 [7,10]. In the following, we concentrate on Hölder regularity and give equivalent conditions on g_n^j; Comparison is made with Sobolev regularity in [6,7,8].

In fact, the number α in (11) is precisely the Hölder regularity order of $\phi(x)$ when $\alpha \leq 1$ [7]:

Theorem 3.1 ([7]) *If $G(1) = 2$, $G(-1) = 0$, and*

$$\max_n |g_{n+1}^j - g_n^j| \leq c\, 2^{-j\alpha} \qquad (13)$$

for some $0 < \alpha \leq 1$, then the binary subdivision scheme converges uniformly to a $\dot{C}^\alpha$ limit function. The converse is true if $\phi(x)$ is stable. In addition, the more regular the limit, the faster the convergence to this limit: For any sequence n_j of integers satisfying (8)*, we have*

$$\sup_x |\phi(x) - g_{n_j}^j| \leq c\, 2^{-j\alpha} \qquad (14)$$

Since we have seen that $\phi(x) \in C^0$ implies $\phi(x) \in \dot{C}^\alpha$ for some $\alpha > 0$, we have the remarkable property that $\phi(x)$ is C^N *if and only if* its Hölder regularity order is greater than N. As pointed out above, this is not true for Sobolev regularity (Fig. 1). Theorem 3.1 also provides a natural graphical interpretation of Hölder regularity in terms of the slopes of g_n^j (12): For example, if these slopes are bounded for all j's, then $\phi(x)$ is $\dot{C}^1$. Finally, Theorem 3.1 gives an interesting indication on the (exponential) rate of convergence of g_n^j towards $\phi(x)$.

For higher Hölder regularity orders, simply consider the derivatives of $\phi(x)$, whose discrete counterparts are the finite differences $\delta^k g_n^j$ (section 2).

Theorem 3.2 **([7])** *If $G(1) = 2$, $G(X)$ has at least $N+1$ zeros at $X = -1$ and*

$$\max_n |\delta^N g_{n+1}^j - \delta^N g_n^j| \leq c\, 2^{-j\alpha} \tag{15}$$

for some $\alpha > 0$, then $\phi(x)$ is $\dot{C}^{N+\alpha}$. The converse is true in the case of stability. Moreover, (15) *implies $\alpha \leq 1$ (if $\phi(x) \not\equiv 0$)*

Note that when the regularity order is greater than $N+1$, α in (15) is necessarily equal to one.

We now present a general algorithm for estimating Hölder regularity. A first difficulty is that Theorem 3.2 provides a test which depends on N: It is only when it turns out that $N < r \leq N+1$ that the criterion is really optimal and provides $N + \alpha = r$. However, the discrete-time characterization of Hölder regularity $N + \alpha$ is equivalent to the same condition in which N and $-\alpha$ have been increased by one [7]. Therefore, by induction, it can be extended to negative values of α (The only restriction is that if α is a nonpositive integer, then $\phi(x) \in \dot{C}^{N+\alpha}$ should be replaced by a slightly weaker condition [7]). Hence, when the criterion gives a *negative regularity order* α, it can be used to prove that $\phi(x)$ has some (positive) regularity if $N > -\alpha$. In particular, if $G(X)$ has no more than $N+1$ zeros at $X = -1$, then Theorem 3.1 necessarily provides the exact regularity order r. In this case, $-\alpha$ is the exact amount of this regularity lost by the "destructive effect," mentionned in section 2, of zeroes in $G(X)$ that are not located at $X = -1$ [8]. This effect typically kills 80% of regularity [1], which explains why the number of zeroes at $X = -1$ in $G(X)$ only gives a weak upper bound, which is not attained unless $G(X)$ has only zeroes at $X = -1$.

To obtain optimal regularity estimates, it is therefore sufficient to estimate α in condition (15). Fortunately, this task can be reduced to a finite number of operations:

Theorem 3.3 **([7])** *Assume $G(1) = 2$, and $G(X)$ has at least $N+1$ zeros at $X = -1$. Let $F_N(X) = 2^{-N} G(X)(1+X)^{-N-1}$ and $\alpha_N = \sup_j \alpha_N^j$, where α_N^j is given by*

$$2^{-j\alpha_N^j} = \max_{0 \leq n \leq 2^j - 1} \sum_k |(f_N^j)_{n+2^j k}| \tag{16}$$

where $(f_N^j)_n$ is defined from $F_N(X)$ similarly as g_n^j.

The sequence α_N^j converges to $\alpha_N \leq 1$ as $j \to \infty$. If there exists j such that $N + \alpha_N^j > 0$, then $\phi(x)$ is $\dot{C}^{N+\alpha_N^j}$ (almost $\dot{C}^{N+\alpha_N^j}$ if $\alpha_N^j \in -\mathrm{N}$ [7]), and, therefore, $\phi(x)$ is $\dot{C}^{N+\alpha_N-\varepsilon}$ for any $\varepsilon > 0$. The regularity estimate is optimal in the case of stability: If $\alpha_N \neq 1$, or if $\alpha_N = 1$ and $G(X)$ has no more than $N+1$ zeros at $X = -1$, then $\phi(x)$ is $\dot{C}^{N+\alpha_N-\varepsilon}$ but is not $\dot{C}^{N+\alpha_N+\varepsilon}$, for any $\varepsilon > 0$.

For a given j and N, $N + \alpha_N^j$ is always a Hölder regularity estimate for $\phi(x)$. This estimate is improved when j increases, and is asymptotically optimal when N is chosen maximal. In practice, the exact (optimal) regularity order r is generally obtained to two decimal places after $j = 20$ iterations [7] (see Fig. 2).

Using a very different approach, Daubechies and Lagarias [4] recently proposed a sophisticated method for estimating Hölder regularity. Their method is easily recovered by rewriting Algorithm 1 in matrix form [7,8], where $2^{-j\alpha_j}$ is estimated by computing spectral radii. While this method is only managable for very short masks (typically of length ≤ 6) and is not always optimal, Theorem 3.3 gives asymptotically optimal results (as j increases) for *any* choice of $G(X)$.

Optimal Sobolev regularity estimates, proposed independently by Cohen and Daubechies [1], can also be derived using our method [6,7,8]. The algorithm simplifies in this case to the computation of the spectral radius of one matrix, but gives suboptimal results as compared with Hölder regularity (see Fig. 2).

4 The Unstable Case

Many optimality results given in this paper fail for "unstable" examples (section 1). In this section, we give a simple trick which allows one to consider another, stable subdivision scheme which has the same regularity properties. As a working example, consider the polynomial mask $G(X) = 2^{-N}(1+X)(1+X^2)^N$. By Theorem 3.1 the limit function $\phi(x)$ exists and is $\dot{C}^1$, hence continuous. The above results cannot improve this regularity order since $G(X)$ has only one zero at $X = -1$. However, $\phi(x)$ is unstable since $1 + X^2$ divides $G(X)$ (see section 1), so we might expect higher regularity for $\phi(x)$. Now consider another mask $\tilde{G}(X) = 2^{-N}(1+X)^{N+1}$. It is easy to see, using Theorem 3.2, that the subdivision scheme $\tilde{g}_n^j$ converges to a $\dot{C}^N$ limit function $\tilde{\phi}(x)$. Since the two masks are related by $(1+X)^N G(X) = (1+X^2)^N \tilde{G}(X)$, it can be easily shown [7,8] that $\phi(x)$ is a linear combination of integer translates of $\tilde{\phi}(x)$. This proves that $\phi(x)$ is $\dot{C}^N$, even though $G(X)$ has only one zero at $X = -1$! It is easy to show [7] that both functions have the same regularity order. Since the regularity estimate $\dot{C}^N$ is optimal for $\tilde{\phi}(x)$ [7], it is also optimal for $\phi(x)$.

Therefore, the argument used in this example has led to an optimal regularity estimate for an *unstable* limit function. This can be easily generalized to the case where unstability is due to the fact that $G(X)$ is divisible by $X^2 - e^{i\omega}$. Note that if the conjecture mentionned in section 1 is true, then this methods works for arbitrary unstable limit functions.

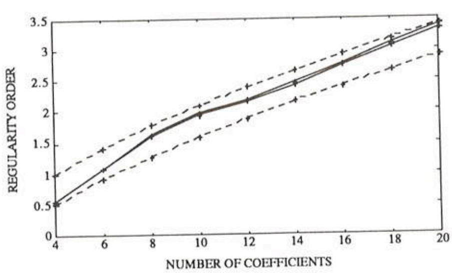

Figure 2: *Regularity estimates for Daubechies wavelets [3] with number of coefficients ranging from 4 to 20. Sobolev lower and upper bounds (dashed). Hölder lower and upper bounds (solid).*

5 A Fast, Sharp Upper Bound for Regularity

It can be shown [6,8] that the computational load of an implementation of (16) is increasing exponentially with j (increasing j by one roughly doubles the number of operations required to compute (16)). Now suppose that (16) is computed only for the values $n = 0$ and $2^j - 1$. The advantage is that the numerical complexity reduces to a linear one in j: we have a much faster algorithm. The price to pay is that α_j is over-estimated: the obtained estimates will only give an *upper bound* for Hölder regularity as $j \to \infty$. The computation of this upper bound can be simplified to the search of the spectral radius of one matrix [7,8]. Although this upper bound seems to be a rough estimation, Fig. 2 shows that the results are very close to be optimal. In fact, Daubechies and Lagarias method [4]—discussed in the preceding section—works if and only if this upper bound turns out to be the optimal Hölder regularity order [7].

Conclusion

This paper has provided a full characterization of regularity in terms of the filter taps. The discrete approach described in this paper is efficient (optimal results are obtained) and inclusive (earlier estimates are recovered). This paper has also provided an easily implementable, optimal Hölder regularity estimation algorithm, which can be used as a tool for quantifying precisely the effect of regularity in practical systems.

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