Fuzzy attribute openings based on a new fuzzy connectivity class. Application to structural recognition in images

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Abstract

In problems such as image segmentation and recognition, the connectivity of target objects is a key feature. In the mathematical morphology framework, connected filters were derived from the classical connectivity theory but do not take into account the imperfections that can affect the image formation. The aim of this paper is twofold: (i) introduce a new class of connectivity for fuzzy objects and (ii) derive some associated attribute openings. We show also that the latters can be performed efficiently using a component-tree representation. We illustrate a potential use of these filters in a brain segmentation and recognition process.

Keywords: fuzzy connected filters, connectivity, hyperconnection, mathematical morphology.

1 Introduction

In image segmentation and recognition, objects of interest are often constrained to be connected. The definition of connectivity depends on the selected representation of objects and the axiomatization of classes of connectivity [9] and of hyperconnectivity [9, 2] provides a rigorous framework to handle the concept of connectivity, which leads to the design of connected filters (e.g. [7]).

In this paper we deal with connectivity of fuzzy objects and associated connected operators, in particular fuzzy attribute openings. Object representation using fuzzy sets theory [11] can lead to more robustness in problems such as image segmentation and recognition. This robustness results to some extent from the partial recovery of the continuity that is lost during the digitization process. The initial definition of fuzzy connectivity [6] provides a crisp characterization of the connectivity of a fuzzy set. Its later extension [2] leads to a characterization of the connectivity as a degree. This degree is however not continuous with respect to the membership function. Therefore we propose a new definition that exhibits better properties, in particular in terms of continuity. Based on this connectivity notion, we define fuzzy attribute openings that we use in a segmentation and recognition context. We show that these operators present nice regularity properties that may lead to more robustness of the process.

We first recall in Section 2 some notations and definitions on fuzzy sets, connections and connected operators. Section 3 deals with fuzzy connectivity notions and an efficient max-tree representation. Section 4 presents a definition of fuzzy attribute openings. In Section 5, we define two practical filters that are illustrated in a recognition process on a brain magnetic resonance image (MRI).

2 Preliminaries

Fuzzy sets – Let X be the digital space $\mathbb{Z}^n$ endowed with a discrete connectivity c_d. A

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fuzzy set on X will be denoted by its membership function $\mu : X \to [0,1]$. We restrict ourselves to fuzzy sets having a bounded support. We denote the α-cut by μ_α and by $\mathcal{F}$ the set of fuzzy sets defined on X. $(\mathcal{F}, \leq)$ is a complete lattice for the usual order on fuzzy sets. The supremum $\vee$ and infimum $\wedge$ are the max and min respectively. The smallest element is denoted by $0_\mathcal{F}$ and the largest element by $1_\mathcal{F}$. As a metric on $\mathcal{F}$ we use: $d_\infty(\mu_A, \mu_B) = \sup_{x \in X} |\mu_A(x) - \mu_B(x)|$, and $(\mathcal{F}, d_\infty)$ is a metric space, inducing a definition of continuity.

Connections and hyperconnections

Definition 1 *[9] Let $(\mathbb{L}, \leq)$ be a lattice. A connected class, or connection, $\mathcal{C}$ is a family of elements of $\mathbb{L}$ such that:*

1. *$0_\mathbb{L} \in \mathcal{C}$,*
2. *$\mathcal{C}$ is sup-generating,*
3. *for any family $\{C_i\}$ of elements of $\mathcal{C}$ such that $\bigwedge_i C_i \neq 0_\mathbb{L}$, then $\bigvee_i C_i \in \mathcal{C}$.*

Let us for instance consider the lattice of subsets of X $(\mathcal{P}(X), \subseteq)$. On this lattice, we use the usual connection $\mathcal{C}_d$ induced by the discrete connectivity c_d (such as 4-connectivity in $\mathbb{Z}^2$). An element of $\mathcal{C}_d$ is then simply a subset A of X that is connected in the sense of c_d (i.e. $\forall (x,y) \in A^2, \exists x_0 = x, x_1, ..., x_n = y, \forall i < n, x_i \in A,$ and $c_d(x_i, x_{i+1}) = 1$).

Connected components of an element A of a lattice $(\mathbb{L}, \leq)$, relatively to a connection $\mathcal{C}$ on $\mathbb{L}$, are the elements C_i of $\mathcal{C}$ such that: $C_i \leq A$ and $\nexists C \in \mathcal{C}, C_i < C \leq A$ (i.e. the largest elements of $\mathcal{C}$ that are smaller than A) [9].

However some connectivities (fuzzy connectivity for instance) cannot be represented by a connection. Dealing with such cases requires to replace the infimum ($\bigwedge$) in condition 3 by another overlap mapping $\perp$ [9], leading to the notion of hyperconnection.

Definition 2 *[9, 2] Let $(\mathbb{L}, \leq)$ be a lattice. A hyperconnection $\mathcal{H}$ is a family of elements of $\mathbb{L}$ such that:*

1. *$0_\mathbb{L} \in \mathcal{H}$,*
2. *$\mathcal{H}$ is sup-generating,*
3. *for any family $\{H_i\}$ of elements of $\mathcal{H}$ such that $\perp_i H_i \neq 0_\mathbb{L}$, then $\bigvee_i H_i \in \mathcal{H}$.*

The hyperconnected components of $A \in \mathbb{L}$ are the elements H_i of $\mathcal{H}$ such that: $H_i \leq A$ and $\nexists H \in \mathcal{H}, H_i < H \leq A$. For any two hyperconnected components H_i and H_j of A, either $H_i = H_j$ or $H_i \perp H_j = 0_\mathbb{L}$. Moreover, $\bigvee_i H_i = A$, where the supremum is taken over all connected components of A.

Connected operators – Connected operators, by definition, manipulate only connected components of the processed image. A generic definition for binary images can be derived from the notion of partition. A partition $P : X \to \mathcal{P}(X)$ satisfies the conditions $\forall x \in X, x \in P(x)$ and $\forall (x,y) \in X^2, P(x) = P(y)$ or $P(x) \cap P(y) = \emptyset$. If $\mathcal{C}$ is a connection on $\mathcal{P}(X)$, the partition is said connected if $\forall x \in X, P(x) \in \mathcal{C}$. We consider in particular the partition $P_\mathcal{C}^A$ which associates a point to the connected component of A or $\bar{A}$ including this point. In addition we say that a partition P_1 is finer than P_2 if $\forall x \in X, P_1(x) \subseteq P_2(x)$.

Definition 3 *[4] An operator $\psi : X \to X$ is connected according to $\mathcal{C}$ if for all $A \subseteq X$ the partition $P_\mathcal{C}^A$ is finer than $P_\mathcal{C}^{\psi(A)}$.*

This definition has been extended to grey-level images considering the largest connected regions that present a constant grey-level. Such operators are known as flat zones filters [8].

However as will be specified next, the connectivity of fuzzy sets is not represented by a connection but by an hyperconnection and the classical definitions of connected operators cannot be applied. A first approach to extend classical connected operators to fuzzy sets is to apply the constraint over all α-cuts.

Definition 4 *An operator ψ defined on $\mathcal{F}$ is connected if $\forall \mu \in \mathcal{F}, \forall \alpha \in [0,1]$, $P_\mathcal{C}^{(\mu)_\alpha}$ is finer than $P_\mathcal{C}^{(\psi(\mu))_\alpha}$.*

3 Connectivity of fuzzy sets

Fuzzy connectivity – The first definition of fuzzy connectivity was proposed by Rosenfeld [6].

Definition 5 *[6] The degree of connectivity between two points x and y of X in a fuzzy set μ ($\mu \in \mathcal{F}$) is defined as:*

$$c^1_\mu(x, y) = \max_{\substack{l \in L_{x,y} \\ l=\{x_0=x, x_1, \ldots, x_n=y\}}} \min_{0 \le i \le n} \mu(x_i)$$

where $L_{x,y}$ denotes the set of digital paths from x to y, according to the underlying digital connectivity c_d defined on X.

Definition 6 *[6] A fuzzy set is said connected if all its α-cuts are connected (in the sense of the connectivity on X).*

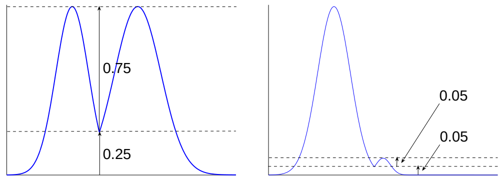
This notion of connectivity is appropriately represented by a hyperconnection on the lattice $(\mathcal{F}, \le)$. We will denote by $\mathcal{H}^1$ the hyperconnection containing all connected fuzzy sets according to Definition 6. It is obtained for the overlap mapping $\perp^1$ defined as [2]: $\perp^1(\{\mu_i\}) = 1$ if $\forall \alpha \in [0,1]$, $\bigcap_i \{(\mu_i)_\alpha \mid (\mu_i)_\alpha \ne \emptyset\} \ne \emptyset$ and 0 otherwise. We denote by $\mathcal{H}^1(\mu)$ the set of hyperconnected components of μ.

An extension to a familly of hyperconnections $\mathcal{H}^1_\tau = \{\mu \in \mathcal{F}, \forall \alpha \le \tau, (\mu)_\alpha \in \mathcal{C}_d\}$ was proposed in [2]. Each $\mathcal{H}^1_\tau$ contains all fuzzy sets whose α-cuts below level τ are connected. The connectivity of a fuzzy set can thus be defined as a degree, instead of a crisp notion, as follows: $c^1(\mu) = \sup\{\tau \in [0,1] \mid \mu \in \mathcal{H}^1_\tau\}$.

As an illustration, the fuzzy sets in Figure 1(a) and (b) have a degree of connectivity of 0.25 and 0.05, respectively. However, intuitively we would rather say that the example in (b) is more connected than the one in (a), which seems to have two very distinct parts. The degree of connectivity depends on the height of the lowest minimum or saddle point, and not on its depth. A small modification in (b) would make the fuzzy set fully connected, illustrating that this definition is not continuous.

A new class of fuzzy connectivity – We now propose another extension of the classical fuzzy connectivity expressed by Definition 6.

Definition 7 *The connectivity degree between two points x and y in a fuzzy set μ is defined by: $c^2_\mu(x, y) = 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, y)$.*



(a) (b)

Figure 1: The degree of connectivity of the fuzzy set in (c) is equal to 0.25, and in (d) to 0.05, although it seems to be more connected.

Definition 8 *The connectivity degree of a fuzzy set μ is defined as: $c^2(\mu) = \min_{(x,y)\in X^2} c^2_\mu(x, y)$.*

It is easy to show that $c^2(\mu)$ is achieved for x such that $\mu(x) = \max_{x' \in X} \mu(x')$, and for y belonging to a regional maximum. Roughly speaking, the connectivity degree of a fuzzy set now depends on the depth of the deepest saddle point in the fuzzy set. On the examples illustrated in Figure 1, it can be observed that the fuzzy set in (a) is 0.25−connected ($1 - 0.75$), while the fuzzy set in (b) is 0.95−connected. In the later case, if one of the modes is progressively shrinking to 0, the degree of connectivity will evolve smoothly towards 1. This is expressed formally by the following result.

Proposition 1 *For fixed x and y, the mapping associating μ to $c^1_\mu(x, y)$ is continuous and Lipschitz, and the mapping associating μ to $c^2_\mu(x, y)$ is continuous and 2-Lipschitz. The mapping associating μ to $c^2(\mu)$ is continuous and 2-Lipschitz.*

Let us now define: $\mathcal{H}^2_\tau = \{\mu \in \mathcal{F} \mid c^2(\mu) \ge \tau\}$.

Proposition 2 *For each $\tau \in [0,1]$, $\mathcal{H}^2_\tau$ defines a hyperconnection.*

We denote by $\mathcal{H}^2_\tau(\mu)$ the set of hyperconnected components of μ and we will speak of τ-hyperconnected components.

Tree representation – From an algorithmical point of view, the obtention of the τ-hyperconnected components and their processing can benefit from an appropriate representation. Since the α-cuts are a core component of our definitions, we can rely on the usual max-tree [7] representation of a function. From now on, we assume that the values of α are quantified, in a uniform way. For each level α of the quantification, nodes of a tree are associated with the connected components (in the sense of $\mathcal{C}_d$) of the α-cut of the considered fuzzy set. Edges are induced by the inclusion relation between connected components for two successive values of α. A fuzzy set μ is then bi-univoquely represented by a tree $T(\mu)$, with:

- $\mathcal{V}$ the set of vertices of the tree (if $v \in \mathcal{V}$, $h(v)$ denotes its altitude, i.e. the value of α corresponding to this node),
- R the root of the tree,
- $\mathcal{L}$ the set of leaves,
- if $v \in \mathcal{V}$, $P^v_{T(\mu)}(h)$ is the subset of $\mathcal{V}$ that contains all the nodes belonging to the chain from the root to v and of altitude less or equal to h.

There are several algorithms for computing the tree, a very recent one being of quasi-linear complexity [5].

Proposition 3 *The set $\mathcal{H}^1(\mu) = \{\mu_i\}$ of 1−hyperconnected components of μ is isomorphic to the set of leaves $\mathcal{L}$, and $T(\mu_i) = P^{l_i}_{T(\mu)}(h(l_i))$, where l_i is the leaf associated with μ_i.*

We denote by $S_{T(\mu)}$ the set of subtrees of $T(\mu)$ which satisfy $\forall S \in S_{T(\mu)}$, $\forall v \in S, P^v_{T(\mu)}(h(v)) \subseteq S$. We define two operators on $S_{T(\mu)}$:
$\varepsilon^r_{T(\mu)}(S) = \bigvee_{l \in \mathcal{L}} P^l_{T(\mu)}(\max(0, h^S_l - r))$,
$\delta^r_{T(\mu)}(S) = \bigvee_{l \in \mathcal{L}} P^l_{T(\mu)}(\min(h(l), h^S_l + r))$.
The first one corresponds intuitively to a contraction of size r of the input subtree S (but it is not rigorously an erosion) and the second one to a dilation of size r.

Proposition 4 *The set of τ−hyperconnected components of a fuzzy set μ is isomorphic to the set of leaves of $\varepsilon^{1-\tau}_{T(\mu)}(T(\mu))$. A τ−hyperconnected component of μ can then be obtained by a dilation of size $(1-\tau)$ of a 1−hyperconnected component of $\varepsilon^{1-\tau}_{T(\mu)}(T(\mu))$.*

Figure 2 illustrates in (b) the component tree $T(\mu)$ of the fuzzy set shown in (a). The 1−hyperconnected component (c) corresponds to one regional maximum of (a), the corresponding subtree is shown in black (b). The results of a contraction of size 0.4 of $T(\mu)$ and the dilation of size 0.4 of one of its connected components are shown in (d) and (e), respectively, providing exactly the sub-tree associated with one 0.6−hyperconnected component of μ (f).

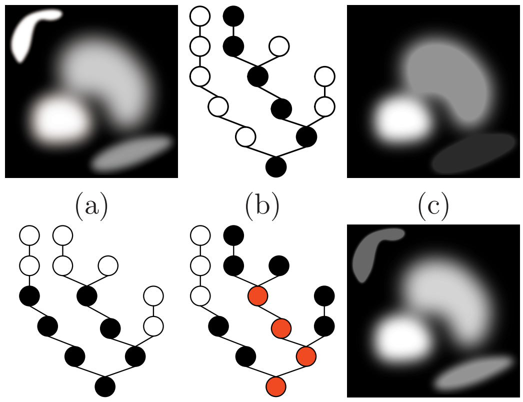

(a) (b) (c)
(d) (e) (f)

Figure 2: (a) Fuzzy set. (b) Component tree (the α-cuts are quantizied with a step 0.2), with a subtree in black corresponding to a 1−hyperconnected component (c). (d) Subtree corresponding to the contraction of size 0.4 (in black). (e) A 0.6−hyperconnected component (in black and red) obtained by dilation of one connected component (in red) of the contraction and the corresponding image (f).

4 Attribute openings applied to fuzzy sets

4.1 Attribute openings based on a crisp criterion

We focus here on segmentation and recognition tasks. In this context we suppose that the aim is to extract a connected object A represented by its membership function μ_A and that a first approximation $\overline{\mu_A} \geq \mu_A$ of this structure (from grey-level prior for instance) is known. Besides we have some prior knowledge about A expressed as a criterion function $f_C : \mathcal{F} \to \{0,1\}$ such that $f_C(\mu_A) = 1$.

The operator on $\mathcal{F}$ defined as:

$$\xi(\overline{\mu_A}) = \bigvee\{\nu \in \mathcal{H}^2_\tau | \nu \leq \overline{\mu_A} \text{ et } f_C(\nu) = 1\}, \tag{1}$$

satisfies the property $\mu_A \leq \xi(\overline{\mu_A}) \leq \overline{\mu_A}$. The resulting fuzzy set is thus a better approximation of μ_A than $\overline{\mu_A}$. We can notice that this operator is increasing, idempotent and anti-extensive and is thus a morphological opening.

However without any condition on f_C the computation of such a filter requires to evaluate the criterion over all elements of $\mathcal{H}^2_\tau$ smaller than $\overline{\mu_A}$ and has an exponential complexity. To overcome this, we can take advantage of the following property: $\forall \nu \in \mathcal{H}^2_\tau, \nu \leq \overline{\mu_A} \Rightarrow \exists \mu_i \in \mathcal{H}^2_\tau(\overline{\mu_A}), \nu \leq \mu_i$. If we restrict ourselves to increasing criteria, the computation of $\xi(\overline{\mu_A})$ can be performed over the τ-hyperconnected components of $\overline{\mu_A}$, the most time consuming operation being the tree computation in quasi-linear time [5]. The filter rewrites:

$$\xi(\overline{\mu_A}) = \bigvee\{\nu \in \mathcal{H}^2_\tau(\overline{\mu_A}) | f_C(\nu) = 1\}. \tag{2}$$

This filter is connected in the sense of definition 4. Moreover it only processes connected components of $\overline{\mu_A}$ and corresponds then to the intuitive notion of attribute openings.

This definition is illustrated in Figure 3. The criterion is defined as a minimal area of 10000 (the area of a fuzzy set is defined as $S(\mu) = \sum_{x \in X} \mu(x)$). During the tree (b) computation from the fuzzy set (a), we compute for each node the associated area. We then obtain the 0.6-hyperconnected components (b) and (c). Their areas are respectively 8612 and 11520 and can be easily obtained from the nodes area. The first one does not satisfy the criterion. The second one does and is the resulting subtree in this case (c) which represents the fuzzy set (d).

4.2 Extension to a fuzzy criterion

The filter proposed in the previous section only manages crisp criteria. It follows that it is not continuous since a small modification of the input set may result in the modification of a complete connected component. To

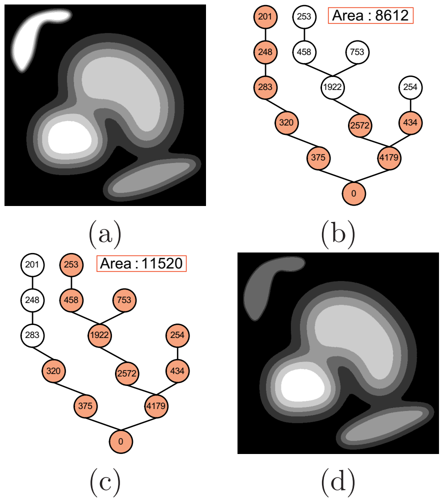


Figure 3: (a) Fuzzy set. Two 0.6-hyperconnected components (b) et (c) in red (the nodes are labeled by their area). The second one satisfies the criterion (c) whereas the first one does not. (d) Resulting fuzzy set.

overcome that and to achieve more robustness in the filtering process, we extend in this section the previous definition to fuzzy criteria. For instance the minimal area criterion can be represented by a membership function (corresponding for instance to a linguistic value such as "large"). The satisfaction of the criterion is thus defined as a degree.

We propose to preserve connected fuzzy subsets whose maximum membership degree is less or equal than the satisfaction degree of the criterion (which guarantees the idempotence of the filter):

$$\xi_{\mu_f}(\overline{\mu_A}) = \bigvee\{\nu \in \mathcal{H}^2_\tau | \nu \leq \overline{\mu_A} \text{ and } \max_{x \in X} \nu(x) \leq \mu_f(\nu)\}. \tag{3}$$

This operator is also a morphological opening and reduces to Equation 1 if μ_f is crisp.

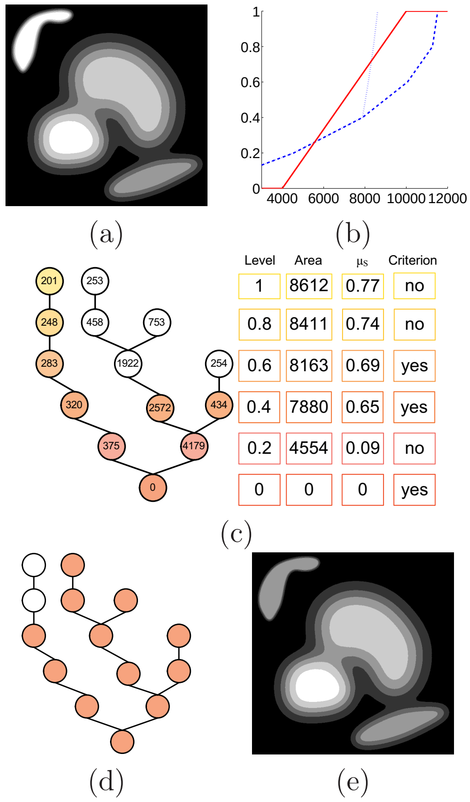
Proposition 5 *If μ_f is continuous and k-Lipschitz, then the mapping associating μ to $\xi_{\mu_f}(\mu)$ is continuous and* $\max(1,k)$*-Lipschitz.*

For computational purposes, we also assume that μ_f is increasing. We can show that Equation 3 then rewrites as:

$$\xi_{\mu_f}(\overline{\mu_A}) = \bigvee_{\mu_i \in \mathcal{H}^2_\tau(\overline{\mu_A})} \bigvee_{m \in [0,1]} \{\min(\mu_i, m) | \\ m \leq \mu_f(\min(\mu_i, m))\}. \quad (4)$$

This leads to a fast computation of $\xi_{\mu_f}(\overline{\mu_A})$ since we only have to handle the τ-hyperconnected components "levelled" at m (i.e. $\min(\mu_i, m)$).

We illustrate this definition in Figure 4. The criterion is here defined as a membership function $\mu_S : \mathbb{R}^+ \to [0,1]$ (b) representing a minimal area. First we extract from the tree (which represents the fuzzy set (a)) the 0.6-hyperconnected components. One is shown in (c). These components are then progressively levelled from 1 to 0 and the satisfaction degree of the criterion $\mu_S(S(\nu))$ is computed for each levelled subtree. If the level is less or equal to this degree we add the levelled subtree to the resulting subtree (d). We show for the 0.6-hyperconnected component in (c) the area at different levels, the satisfaction degree of the criterion μ_S and finally whether the levelled subtree has to be added to the result or not.



Level	Area	μ_S	Criterion
1	8612	0.77	no
0.8	8411	0.74	no
0.6	8163	0.69	yes
0.4	7880	0.65	yes
0.2	4554	0.09	no
0	0	0	yes

Figure 4: (a) Fuzzy set. (b) μ_S in red plain, in dashed blue the values $(S(\min(\mu_i, m)), m)$. One 0.6-hyperconnected component (c). For each level m, we show the area of $\min(\mu_i, m)$, $\mu_S(S(\min(\mu_i, m)))$ and the satisfaction of the criterion $\max_{x \in X} \nu(x) \leq \mu_S(S(\nu))$. Resulting subtree (d) and associated fuzzy set (e).

5 Filtering

We propose in this section two connected filters that can be used in a segmentation and recognition process, implementing the idea of deriving an estimation of μ_A from a first rough overestimation $\overline{\mu_A}$ and a criterion.

5.1 Marker inclusion based

We define a criterion from another estimation $\underline{\mu_A}$ of μ_A, such that $\underline{\mu_A} \leq \mu_A$ ($\underline{\mu_A}$ is a marker of the target object):

$$\xi^1_{\underline{\mu_A}}(\overline{\mu_A}) = \bigvee \{\nu \in \mathcal{H}^2_\tau | \nu \leq \overline{\mu_A} \text{ and } \underline{\mu_A} \leq \nu\}. \quad (5)$$

However, as illustrated in Figure 5, this filter is not continuous with respect to $\underline{\mu_A}$. The marker in dashed red (b) satisfies the inclusion constraint with two connected components of (a). A small modification of this marker leads in (c) to the satisfaction of the constraint with the four connected components. In (d) the included set is not strictly included in any connected component of (a). The filter thus returns an empty set.

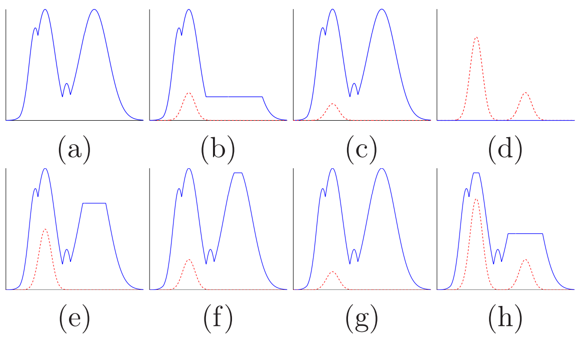


Figure 5: Filtering of a fuzzy set (a) by various markers (in red) according to Equation 5 (b-d) or Equation 6 (e-h). The result is displayed in blue.

Instead of considering a strict inclusion, we can rely on a fuzzy one, based on Lukasiewicz operator [3]: $\mu_{\leq}(\mu_A, \mu_B) = \min_{x \in X} \min(1, 1 - \mu_A(x) + \mu_B(x))$. The filter defined by Equation 3 then writes:

$$\xi^2_{\underline{\mu_A}}(\overline{\mu_A}) = \bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \overline{\mu_A} \text{ and } \max_{x \in X} \nu(x) \leq \mu_{\leq}(\underline{\mu_A}, \nu)\}. \quad (6)$$

Proposition 6 *Let $\alpha = \max_{x \in X} \underline{\mu_A}(x)$. The result of the connected filter defined in Equation 5 is $(\alpha - (1 - \tau))-$hyperconnected.*

The results of this filter are also illustrated in Figure 5 (e-h). We can notice that the input fuzzy set is now progressively filtered when the marker gets larger and larger. Intuitively, hyperconnected components verifying the inclusion constraint are kept, while the other ones are reduced to a level corresponding to the degree of satisfaction of the constraint.

Proposition 7 *The mapping associating $\underline{\mu_A}$ to $\xi^2_{\underline{\mu_A}}(\overline{\mu_A})$ is continuous and Lipschitz, such as the mapping that associates $\overline{\mu_A}$ to $\xi^2_{\underline{\mu_A}}(\overline{\mu_A})$.*

We illustrate now this connected filter on a brain recognition task in Figure 6. As an example, we want to extract the right lateral ventricle from a brain MRI (a). We rely on anatomical knowledge expressed as spatial relations between structures [1], and on grey level information. This allows us to obtain an over-estimation $\overline{\mu_{LVr}}$ as close as possible to the structure of interest. We define $\overline{\mu_{Gl_{LVr}}}$ representing the knowledge on grey levels, so as to have $\mu_{LVr} \leq \overline{\mu_{Gl_{LVr}}}$ (b). Once the brain has been segmented, it becomes possible to represent the central location of the ventricles inside the brain (c), so as to guarantee $\mu_{LVr} \leq \overline{\mu_{Sp_{LVr}}}$. The conjunctive fusion of $\overline{\mu_{Sp_{LVr}}}$ and $\overline{\mu_{Gl_{LVr}}}$ is shown in (d), and provides an over-estimation $\overline{\mu_{LVr}}$. Although the over-estimation has been strongly reduced, it still exhibits several connected components. We illustrate now the effect of $\xi^2_{\underline{\mu_{LVr}}}(\overline{\mu_{LVr}})$, based on a marker $\underline{\mu_{LVr}}$ defined as a fuzzy set having a support reduced to one point centered in the right lateral ventricle, with a membership value taking values 1 (which mostly selects the right ventricle), 0.75, 0.5 and 0 (which does not filter), respectively (e–h). A potential application of this approach is to perform a filter, preserving connectivity properties, and being more or less strong depending on the confidence we may have in the marker.

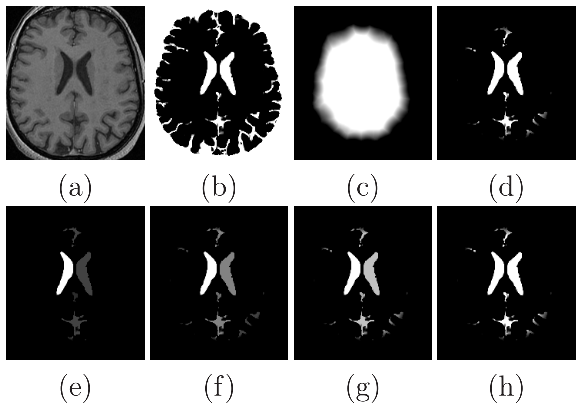


Figure 6: (a) One axial slice of a 3D brain MRI. (b) Grey level information: $\overline{\mu_{Gl_{LVr}}}$. (c) Central location inside the brain: $\overline{\mu_{Sp_{LVr}}}$. (d) Conjunctive fusion. (e–h) Results of the connected filter specified by Equation 6 using a marker centered in the right ventricle, with maximal value 1, 0.75, 0.5, 0, respectively.

5.2 Fuzzy area opening

Area opening is one of the well known connected operators [10]. It filters connected components over a minimal area criterion, and can be formulated as:

$$\xi_{S_{\min}}(A) = \bigvee\{c \in \mathcal{C} | c \leq A \text{ et } S(c) \geq S_{\min}\},$$

where $\mathcal{C}$ is a connection over X and S a function that returns the area. In the example developed above we used a criterion based on the area of fuzzy objects. However it may be more appropriate to consider a fuzzy measure [3]: $\mu_S(\mu)(v) = \sup_{S(\mu_\alpha) \geq v} \alpha$. We can notice that this membership function is decreasing. If we consider as a criterion a minimal area $S_{\min}$ we obtain:

$$\xi^1_{S_{\min}}(\mu_A) = \bigvee\{\nu \in \mathcal{H}^2_\tau | \nu \leq \mu_A \text{ and } \max_{x \in X} \nu(x) \leq \mu_{S(\nu)}(S_{\min})\}.$$

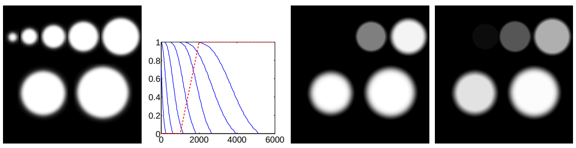
For more flexibility in recognition tasks, it is more appropriate to represent the criterion by a membership function $\mu_{S_{\min}} : \mathbb{R}^+ \rightarrow [0, 1]$

(for instance a ramp function replacing the crisp threshold). The filter then rewrites:

$$\xi^2_{\mu_{S_{\min}}}(\mu_A) = \bigvee\{\nu \in \mathcal{H}^2_\tau | \nu \leq \mu_A \text{ and}$$
$$\max_{x \in X} \nu(x) \leq \max_{v \in \mathbb{R}^+} \min(\mu_S(\nu)(v), \mu_{S_{\min}}(v))\}.$$

Figure 7 illustrates these filters. The fuzzy set (a) contains 7 objects of increasing area. Their fuzzy area $\mu_S(\mu)(v)$ is represented in (b). We first apply ξ^2_{1202} (c). All α-cuts of the 1-hyperconnected components that satisfy the criterion are selected. The use of a membership function (b) as criterion leads to more robustness of the filter (d).

Proposition 8 *The mapping that associates μ to $\xi^2_{\mu_{S_{\min}}}(\mu)$ is continuous and Lipschitz, as well as the mapping that associates $\mu_{S_{\min}}$ to $\xi^2_{\mu_{S_{\min}}}(\mu)$.*



(a) (b) (c) (d)

Figure 7: (a) Fuzzy set that contains 7 objects. (b) $\mu_S(\mu)(v)$ for each object in blue and $\mu_{S_{\min}}$ in dashed red. (c) ξ^1_{1202}. (d) $\xi^2_{\mu_{S_{\min}}}$.

Figure 8 illustrates this filter on a brain MRI example. We filter an overestimation $\overline{\mu_{LV}}$ of the lateral ventricles (b) according to a minimal volume prior represented by the membership function $\mu_{S_{\min}}$. The resulting fuzzy subset $\xi^2_{\mu_{S_{\min}}}$ is shown in (c). Some components corresponding in particular to the sulci are efficiently removed and we thus obtain a better approximation of the lateral ventricles, which can serve as a very good initialization for a precise segmentation process.

6 Conclusion

In this paper we have introduced a new definition of connectivity for fuzzy objects and associated fuzzy attribute openings. This connectivity tends to overcome some drawbacks of classical definitions. We have shown that the associated attribute openings exhibit some nice continuity properties that are of prime importance in image segmentation and recognition tasks. Two specific attribute openings were presented and illustrated as a component of a recognition process of brain MRI.

(a) (b) (c)

Figure 8: (a) One axial slice. (b) $\overline{\mu_{LV}}$. (c) $\xi^2_{\mu_{S_{\min}}}(\overline{\mu_{LV}})$.

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