A New Definition for Fuzzy Attributed Graph Homomorphism with Application to Structural Shape Recognition in Brain Imaging

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Abstract

We propose in this paper an original definition for fuzzy attributed graph homomorphism that deals with both the structural aspect and the maximization of similarity between attributes (on nodes and on arcs). A graph fuzzy homomorphism theory is developed to gather different pieces of information extracted from images, and to take full advantage of the whole graph structure, with several attached attributes. The homomorphism properties allow relevant and accurate graph mapping by evaluating graph structure deformation with fuzzy sets. Anticipated application is atlas-based labeling of brain MRI using graph homomorphism for spatial and temporal analysis of brain tumors and surrounding structures. The definitions and properties proposed in this paper are a first step towards this goal.

1. Introduction

Graph modeling is being used in structural pattern recognition for several years. The graph structure can easily model adjacency between objects in a scene (classical 2D images, or 2D image sequences, or 3D images, or 3D image sequences). In this case, each vertex represents an individual object and an arc represents a binary relation (adjacency is but one example of such relations) between two objects. Recent works ([6, 12] for example) used a graph structure to model relations and attached attributes on both vertices and arcs to quantify the objects and the relations. These attributes are extracted from the image and are supposed to be relevant for recognition purposes. Classical work is then to look for similarities between an unknown graph and several model graphs. Well known examples are problems of character recognition (e.g. in [9], Chinese character are modeled with fuzzy attributed graphs) or face recognition (e.g. [13]). A graph structure can also model relations between object classes. For example, in [8], relationships between classes extracted on an image with a fuzzy c-means [11] are modeled. Often, the problem is to find between several models which one fits the case that is analyzed. The problem involved is known as graph isomorphism: a one-to-one mapping is searched. This problem is NP-hard.

In complex scene recognition, there is hardly a unique and exact solution. Previous processing on the scene can infer segmentation problems that imply imprecision and uncertainty. Moreover, another type of problems is not to find the model among a set of models that best fits the image but to find a more precise mapping between each region of the image (vertices in the graph) and labels of one generic model. Our example is the recognition of pathological structures in brain Magnetic Resonance Images (MRI). A "normal brain" is manually segmented by radiologists and is used as an atlas for recognition. This constitutes the generic model. Another image is automatically segmented. Two graphs are built: one from the atlas, and another from the image to process. If we want at least all the boundaries of the anatomical brain structures, the algorithm has to produce over-segmentation. The use of bijective graph matching (isomorphism) is therefore inadequate, and several-to-one or one-to-several mappings of objects must be investigated. Unfortunately, the literature on the subject does not contain a unified and generic definition of such graph homomorphism.

The contribution of this paper is the proposal of an original and generic definition of a fuzzy graph homomorphism (also called morphism) that processes both vertices and arcs. We also introduce measuring tools for fuzzy attributed graphs. The proposed definition is then applied to a concrete brain structure recognition problem in medical imaging, which is another original aspect of our contribution.

After preliminary notations exposed in Section 2, this paper briefly explains the use of fuzzy attributed graphs (Section 3). The definition of a fuzzy graph homomorphism is presented in Section 4. Measuring tools for fuzzy attributed graphs are explained in Section 5. Finally a discussion on a first application (Section 6) and a conclusion (Section 7) are given.

2. Notations

We need two graph structures to define the morphism: $G_i = (N_i, E_i)$, where N_i is a vertex set, $E_i \subseteq N_i \times N_i$ is an arc set, and $i \in \{1, 2\}$ refers to the graphs.

Let X be a set and $\rho : X \to [0, 1]$ a fuzzy set on X. The application ρ is called the membership function to the set X. The support of ρ is the subset of X: $supp(\rho) = \{x \in X / \rho(x) > 0\}$.

For all sets X, we denote by $F(X)$ the set of all the fuzzy subsets on X. Specifically, $\rho \in F(X)$.

3. Fuzzy attributed graphs

We need first to define fuzzy attributed graphs (FAG) and their use in structural pattern recognition. Previous similar definitions can be found in [6, 8, 9]. Let $Z = \{z_i\}_{i=1,\dots,I}$ be the vertex attribute set. Values for each attribute z_i is taken in the set $R_i = \{r_{ij}\}_{j=1,\dots,J_i}$. A fuzzy set is introduced on these values: $\tilde{V}_{R_i} \in F(R_i)$. Finally, the set $\tilde{L}_v = \{(z_i, \tilde{V}_{R_i})\}_{i=1,\dots,I}$ stands for the vertex attributes and fuzzy set pairs. We define similar sets for the arcs. Let $Y = \{y_i\}_{i=1,\dots,I'}$ be the arc attribute set. Values for each attribute y_i is taken in the set $T_i = \{t_{ij}\}_{j=1,\dots,J'_i}$. A fuzzy set is introduced on these values: $\tilde{A}_{T_i} \in F(T_i)$. Finally, the set $\tilde{L}_a = \{\tilde{A}_{T_i}\}_{i=1,\dots,I'}$ stands for the arc attributes and fuzzy set pairs.

A fuzzy attributed graph on $\tilde{L} = (\tilde{L}_v, \tilde{L}_{\tilde{a}})$ with an underlying graph structure $G_i = (N_i, E_i)$ is the ordered pair $(\tilde{V}, \tilde{A})$ where $\tilde{V} = (N, \tilde{\sigma})$ is called the fuzzy attribute vertex set and $\tilde{A} = (E, \tilde{\mu})$ the fuzzy attribute arc set.

$$\tilde{\sigma} : N \to \tilde{L}_v \qquad \text{is the fuzzy vertex interpreter}$$
$$\tilde{\mu} : E \to \tilde{L}_a \qquad \text{is the fuzzy arc interpreter}$$

This definition can also handle other graph types such as simple graphs, some weighted graphs, fuzzy graphs, attributed graphs. The simple graph is just the structure of the FAG: $Z = Y = \emptyset$. The weighted graph can be handled with $Z = \emptyset$ and Y holds one attribute: the normalized weight (if no weight is infinite). The fuzzy graph [10] is handled with Z and Y holding one attribute: the membership function of the vertex, or the arc to the graph. The attributed graph is the crisp version of the FAG, where all the membership functions take their values in $\{0, 1\}$ instead of $[0, 1]$. We aim to work on FAG because of its generic property, and also because of the richness of this formalism in pattern recognition problems, particularly in the image domain.

4. Fuzzy homomorphism

The definition of the graph homomorphism is defined on a simple graph structure, and is extended to fuzzy attributed graphs with the introduction of several tools.

A fuzzy homomorphism between the graph G_1 and the graph G_2 is the pair of membership functions (ρ_σ, ρ_μ) defined on the vertices and on the arcs:

$$\rho_\sigma = \{\rho_\sigma{}^{u_1} : N_2 \to [0, 1],\ u_1 \in N_1\}$$
$$\rho_\mu = \{\rho_\mu{}^{(u_1,v_1)} : E_2 \to [0, 1],\ (u_1, v_1) \in E_1\}$$

such that $\forall (u_1, v_1) \in E_1$,

$$\rho_\mu{}^{(u_1,v_1)}(u_2, v_2) > 0 \Rightarrow \rho_\sigma{}^{u_1}(u_2) > 0 \text{ and } \rho_\sigma{}^{v_1}(v_2) > 0 \qquad (1)$$

$$\forall (u_1, v_1) \in E_1 : supp(\rho_\sigma{}^{u_1}) \neq \emptyset \text{ and } supp(\rho_\sigma{}^{v_1}) \neq \emptyset$$
$$\Rightarrow \exists (u_2, v_2) \in supp(\rho_\sigma{}^{u_1}) \times supp(\rho_\sigma{}^{v_1})$$
$$\text{such that } (u_2, v_2) \in supp(\rho_\mu{}^{(u_1,v_1)}). \qquad (2)$$

The aim of this definition is to work on both the vertices and the arcs as a whole structure, whereas other definitions work only on vertices and consider the arcs as a help to match two graphs. Thus, the two properties (1) and (2) are given to keep graph structures. That is why there are relationships between vertex and arc membership functions. The property (1) is introduced to guarantee that an arc has two vertices at its extremities: this constraint is the basic property of a graph. Here, this constraint is expressed in terms of degrees. Alone, this property allows the correspondence by the morphism between two vertices linked with an arc to others two without arc. To explain the introduction of Equation 2, let's first consider the reciprocal property that we call the bijective condition (Equation (3)).

$$\forall (u_1, v_1) \in E_1,\ \rho_\mu{}^{(u_1,v_1)}(u_2, v_2) > 0$$
$$\Leftrightarrow \rho_\sigma{}^{u_1}(u_2) > 0 \text{ et } \rho_\sigma{}^{v_1}(v_2) > 0 \qquad (3)$$

This condition is too strong for a generic problem. If two vertices have the same vertex image by the morphism, then the arc between them must have a loop on this vertex. In Figure 1, the vertex u and the vertex v are linked to the vertex w with a non zero degree (symbolized with a dashed line). If the condition (3) is verified, the arc (u, v) must be linked to the loop (w, w). Loops are very often meaningless in pattern recognition problems modeled by graphs. Thus, this condition cannot be kept for a generic definition.

Because the condition (1) could alone imply trivial morphisms that only match vertices, another original property has been introduced. This property, formalized by Equation (2), is a weak reciprocal condition to (1). If two vertices

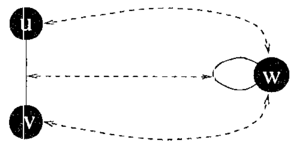


Figure 1. Loop problem with the too strong bijective condition (3).

linked with an arc have several vertex images by the morphism, then there must exist at least one arc between the vertex images that is the image of the first arc. This condition is illustrated in Figure 2. We have $\rho_\sigma{}^u(x) \neq 0$ with $x \in \{a, b, c\}$. We have $\rho_\sigma{}^v(y) \neq 0$ with $y \in \{d, e, f, g\}$. These properties are symbolized with dashed lines between the vertices. Then, there exists at least one arc between one vertex in $\{a, b, c\}$ and another in $\{d, e, f, g\}$ that is linked to (u, v): $\rho_\mu{}^{(u,v)}((b, f)) \neq 0$. This morphism satisfies both conditions (1) and (2).

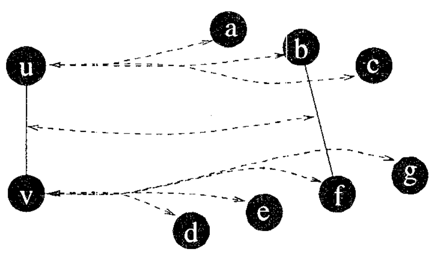


Figure 2. The weak reciprocal condition (2) guarantee that the support of the membership function of an arc is not empty.

This condition allow the potential conservation of the graph structure. The problem of Figure 1 is solved on Figure 3: other nodes $\{w, x, y, z\}$ can be matched to u and v. Only one arc between one node of (w, x) and one node of $\{w, y, z\}$ (that is not the arc (w, w)) is sufficient. Two arcs are present on Figure 3 to show two possibilities in this case. On the one hand, the arc (x, z) is a normal arc that can show that the morphism on both node u and v to the node w was not significant. On the other hand, the arc (w, y) show that the morphism of both node u and v to the node w can be significant, but that a graph structure can however be kept with the arc (w, y). The final step of defuzzification, that is the decision step, will be able to choose to match this arc, or not. Keeping the arc can be really helpful during the algorithmic research of the morphism.

It is easy to show that the condition (3) implies the conditions (1) and (2) but that the reciprocal is false. This formalism allows both several-to-one and one-to-several ap-

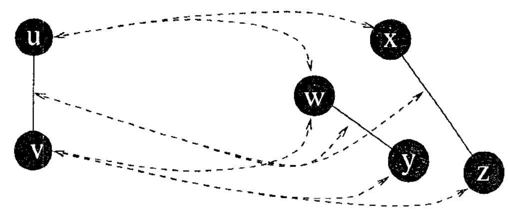


Figure 3. Loop problem solved by the weak reciprocal condition (2).

plications, and conservation of the graph structure. On the one hand, this property induces more complicated treatment when optimizing the morphism, but on the other hand it can handle segmentation problems such as under-segmentation and over-segmentation that are classical drawbacks of image segmentation.

5. Similarity measures

Similarity measures defined in our framework are based upon the work of Rifqi et al [5] who proposed a whole theory on comparison measures of fuzzy objects. The measures are extended to fuzzy attributed graphs with several attributes. The reader should refer to this article for further explanation of the theory. An attribute is a set of different fuzzy sets. Each arc and each vertex have several attributes. In order to compare two vertices (or two arcs) with the same set of attributes, we have to define aggregation operators.

Let $\tilde{\sigma}_i(u)$ be the fuzzy vertex attribute interpretor for attribute i, on the vertex u. We also define $\tilde{\mu}_j(u, v)$, the fuzzy arc attribute interpretor for attribute j, on the arc (u, v). We will only define the measures for the vertices: the measures on the arcs are defined similarly. A vertex u has the following indexed vector of fuzzy set attributes: $(\tilde{\sigma}_1(u), \tilde{\sigma}_2(u), \dots, \tilde{\sigma}_I(u))$, with I the number of attributes.

A $\mathcal{M}$-measure $\mathcal{M}_{\tilde{\sigma}}$ on a vertex u is defined by the function:

$$\begin{aligned} \mathcal{M}_{\tilde{\sigma}} : N \rightarrow &\quad [0, 1]^I \\ u \mapsto &\quad \mathcal{M}_{\tilde{\sigma}}(u) \end{aligned}$$

with

$$\mathcal{M}_{\tilde{\sigma}}(u) = (M_1(\tilde{\sigma}_1(u)), M_2(\tilde{\sigma}_2(u)), \dots, M_I(\tilde{\sigma}_I(u)))$$

with M_k being a simple measure on a single fuzzy set (monotonous function, with $M(\emptyset) = 0$). The first property is that $\mathcal{M}_{\tilde{\sigma}}$ is also monotonous with respect to the inclusion (the proof is immediate). For two vertices u and v, we note: $\mathcal{M}_{\tilde{\sigma}}(u) \preceq \mathcal{M}_{\tilde{\sigma}}(v)$.

A M_i-measure of similitude between two fuzzy set A and B is a mapping in $[0,1]$ such that $M_i(A,B) = G_{M_i}(A \cap B, A - B, B - A)$, and that is non-decreasing in $A \cap B$ and non-increasing in $A - B$ and $B - A$. Justifications can be found in [5].

Let $S^i_{\tilde{\sigma}}$ be an M_i-measure of similitude on a fuzzy set $\tilde{V}_{R_i}$ ($S^i_{\tilde{\sigma}}$ is thoroughly defined in [5], but the complete definition is not necessary to understand the concept presented here). The application defined by:

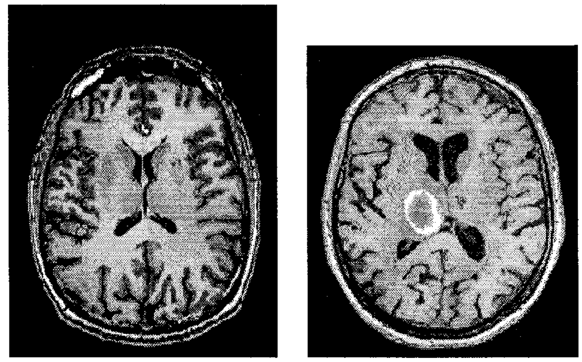
$$S_{\tilde{\sigma}} : N_1 \times N_2 \to [0,1]$$
$$(u_1, u_2) \mapsto \mathcal{T}(S^1_{\tilde{\sigma}}(\tilde{\sigma}_1(u_1), \tilde{\sigma}_1(u_2)), \ldots, S^I_{\tilde{\sigma}}(\tilde{\sigma}_I(u_1), \tilde{\sigma}_I(u_2))),$$

with $\mathcal{T}$ a fusion operator chosen for example from [1], like a t-norm or a t-conorm. We can prove that $S_{\tilde{\sigma}}$ is therefore an $\mathcal{M}$-measure of similitude. We have defined a global measure that can be applied on vertices, and that has the same global properties [5] than the individual single fuzzy set measure of similitude. All these definitions can be applied to the arc attribute sets to build $S_{\tilde{\mu}}$, an $\mathcal{M}$-measure of similitude.

6. First application and discussion

The aim of the homomorphism definition is to be generic. Applications are indeed numerous, and we will only present our first application of it in brain medical imaging. Brain anatomical structures are visible on Magnetic Resonance Images. This medium is also widely used by radiologist to visualize brain tumors, after injection of gadolinium (a contrast product). Segmentation and pattern recognition issues are important for such images. The tumor itself must be segmented and recognized in order to measure its properties such as volume or radiometry. These characteristics can be measures through time with periodic regular acquisition, to control their evolution. Those problems are important and already studied although no real satisfactory solutions were found yet. Another issue not studied yet is the segmentation of the anatomical structures surrounding the tumor, and that are distorted, necrosed, or simply no longer exist. The aim of such a study is to measure the effect of a therapy on both the tumor and the brain structures. Therapy are mainly of three types: chimiotherapy, radiotherapy or surgical intervention. There are two examples of MR images in Figure 4. Figure 4-a represents an axial slice of a 3D normal brain, and Figure 4-b represents an axial slice (approximately corresponding to 4-a) of a 3D brain with a tumor (white hyper-signal).

To achieve this, we aim at developing a robust atlas-guided labeling of over-segmented images using fuzzy attributed graphs. We first use an anatomical atlas: a "normal" brain MRI is manually segmented by a radiologist. This segmentation is used afterwards as an atlas. A fuzzy attributed graph is then built on this atlas. We compute an automatic segmentation on a pathological brain MRI, after a preliminary segmentation of the tumor itself (we only want surrounding structures to be recognized). If we want at least all the anatomical structures boundaries, the chosen algorithm produces over-segmentation that must be handled in our algorithm. We build fuzzy attributed graph on this image. The over-segmentation forbid the use of isomorphism between the two graphs. The theory presented in section 4 is the only known definition of graph morphism that can handle other mappings than one-to-one. That is why we developed and used it.



(a) (b)

Figure 4. MR image examples: (a) is an axial slice of a normal brain, (b) is an axial slice of a pathological brain with a tumor.

The chosen attributes depend upon the way we are building both graphs. Because spatial relations are scarcely variable in brain MRI, the graph structure was build upon an adjacency graph: one vertex corresponds to one region, and one arc to the adjacency of two regions. Thus, a first natural fuzzy attribute for the arcs were a fuzzy adjacency: this measure is used because it is directly linked with the construction of the graph. We also introduced a fuzzy distance that can overcome short range influence of adjacency (we refer to [3, 4] for further explanations of fuzzy adjacency and fuzzy distance). We also choose an attribute about fuzzy relative position [2] because adjacency and distances do not express 3D directional spatial relations. The vertex attributes are somewhat more difficult because of the variability of the shape, absolute position and grey levels of the brain structures. Therefore, we can foresee that the vertex attributes should not have an important weight in the algorithm that will find the morphism. The only attribute tested was the membership degree to the three substance classes of the brain: grey matter, white matter and cerebro-spinal fluid. This attribute is processed from the grey levels and a-priori knowledge of the statistical properties of them.

We have adapted a fuzzy relaxation algorithm proposed originally by Rosenfeld [11] and modified by Chipman et al. [7] to overcome the problems of contagious low weights. This algorithm was not proposed for graph matching, but the formalization was very close. Our first adaptation uses only the morphism ρ_σ on the vertex, and consider the morphism ρ_μ on the arcs only as a fuzzy compatibility between the labeling of two pairs of nodes. The algorithm initializes ρ_σ (after a rigid matching of the two MRI volumes) by comparing the center of gravity of each region (i.e. vertex) to the center of gravity of each region of the atlas. This operation is possible because both volumes have the same orientation and scaling. The algorithm maximizes the following criterion:

$$J = \sum_{u_1, v_1, u_2, v_2} S_{\bar{\mu}}((u_1, v_1), (u_2, v_2)) \rho_\sigma(u_1, u_2) \rho_\sigma(v_1, v_2), \tag{4}$$

with $u_1 \in N_1$, $v_1 \in N_1$, $u_2 \in N_2$, $v_2 \in N_2$. In this criterion, $S_{\bar{\mu}}((u_1, v_1), (u_2, v_2))$ is in fact a compatibility coefficient computed once for each pair $((u_1, v_1), (u_2, v_2))$ and kept constant during the algorithm.

First results have been obtained, and are very promising. One typical result is shown in Figure 5. All the images are 3D, and this figure represents two slices (axial and sagittal) of the same 3D image. Figures 5-a and 5-b are the atlas labels represented as grey levels (one grey level equals one anatomical structure, and there are 50 structures). Figures 5-c and 5-d show an example of an over-segmentation of a MR image that we want to analyze. There are 400 regions in this 3D segmentation. Finally, Figures 5-e and 5-f are the results of labeling (by our algorithm) superimposed on the original MR images. We can see that the two main regions recognized as the caudate nucleus fit well the dark grey structure. The result of the morphism is in fact several fuzzy sets that have to be interpreted. Those first results were obtained by selecting the first regions that best fit a given structure. Only the number of selected regions is chosen in a supervised way in our current implementation.

The results are promising, although not yet thoroughly studied. The fact that there is no gold standard to compare our result with is a serious difficulty to the evaluation of the results. One way is to compare manual segmentation and recognition of the structure. This method, which takes a long time, is not yet considered because all the features of our new graph morphism are not exploited. The next issue is to match both vertices and arcs, and not only vertices. We have to create a new relaxation algorithm, based upon the one used for this test.

The first tests were on normal brains. Tests on pathological ones are planned, and are supposed to show the robustness of the method with respect to more important deformations, because of the use of relative spatial attributes in the graph. These results will be available soon.

7. Conclusion

We have presented a new generic definition of graph fuzzy homomorphism that should include all previous definitions as particular cases. Minimal properties have been proposed and discussed in order to keep graph structure through the morphism. Several properties of this defini-

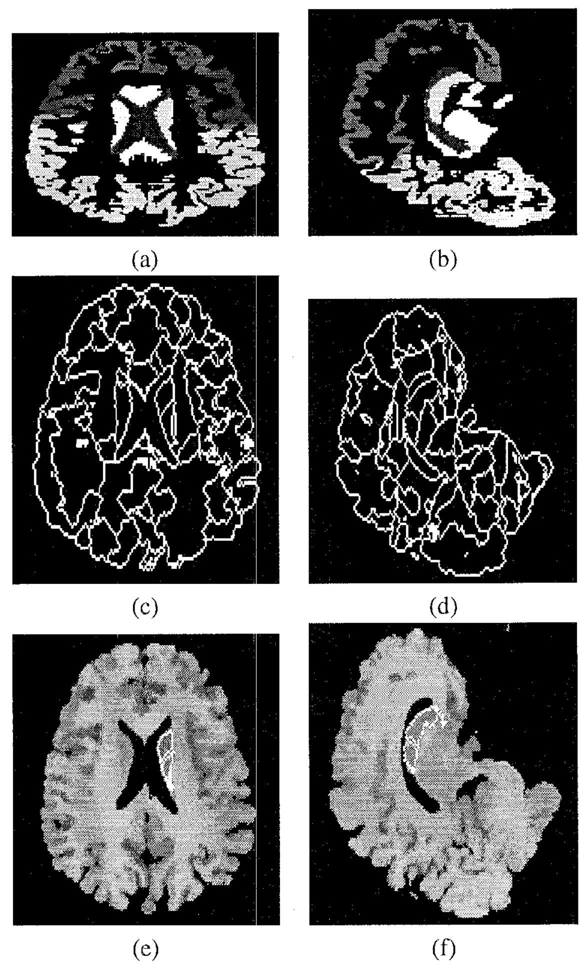


Figure 5. Example of segmentation and recognition result: (a) and (b) are two atlas slices, (c) and (d) are two slices of an over-segmentation example of a normal brain, (e) and (f) are two results of a recognized structure (caudate nucleus) superimposed on the original MR image.

tion are not explored yet (composition, reciprocal morphism ...). A methodology to build fuzzy measures on fuzzy attributed graphs is also proposed, and aims at defining tools to measure the accuracy of the matching. Both theories have been applied to the problem of brain MR image segmentation and recognition for anatomical structures. First results are promising. A complete evaluation is the scope of future work. The problem of graph morphism finding was not explored in this paper, and will soon be the subject of further work.

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