INFERENCE OF DIRECTIONAL SPATIAL RELATIONSHIP BETWEEN POINTS: A PROBABILISTIC APPROACH

Sidi Mohammed Réda Dehak, Isabelle Bloch, Henri Maître

Ecole Nationale Supérieure des Télécommunications
Département TSI - CNRS URA 820, 46 rue Barrault, 75013 Paris, France
Tel: 33 1 45 81 76 36, Fax: 33 1 45 81 37 94, E-mail: dehak@tsi.enst.fr

ABSTRACT

This paper develops an evaluation of the position probability of a point C which is known to be in a direction β with respect to a point B, itself in the direction α with respect to another point A. The obtained results can be used in the problem of inference of directional relationships in the case of spatial reasoning.

1. INTRODUCTION

Spatial relations between objects or regions in images play an important role in many fields, like pattern recognition, computer vision, scene interpretation, Geographical Information Systems (*GIS*), scene descriptions in natural language and autonomous navigation of mobile robots. An important part of the spatial relationships between objects can be described in terms of directional spatial relationships, like "object B is *to the North-East of* object A". Many previous works addressed the problem of defining and evaluating spatial relationships, an important part of which uses fuzzy approaches [1, 2, 3, 4].

Often the complexity of the algorithms to evaluate various spatial relationships does not encourage the exploitation of such information, especially in real time applications such as autonomous navigation. In order to reduce the evaluation cost, it is possible to exploit the properties of transitivity, symmetry, etc. of these relations to deduce new spatial relations from those already evaluated. This deduction or inference of spatial information can be seen under various points of view [5]. For instance we may:

1. deduce spatial relationships between two objects A and C knowing those connecting them to another object B,
2. deduce complex spatial relations from simple ones,
3. deduce spatial relations between moving objects at time t knowing those at time t_0.

In this paper, we are interested in the first problem. More precisely, we answer the following question: if we know that the object B is to the North-East of the object A and the object C is to the North-West of the object B, what can be said about object A and object C using only directional spatial relationships?

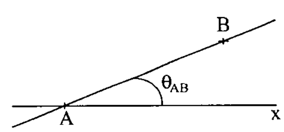
This approach can be used to reduce index size in *GIS* database by storing only principal spatial relations and use inference algorithms to obtain the others. We can also reduce the time to evaluate spatial relations in autonomous navigation of mobile robots. For instance if a robot A sees a landmark B and can evaluate its position with respect to it, it can infer its position with respect to a landmark C from the knowledge of the relative position between the two landmarks, as well as the evolution of this position in time.

The problem of representation of spatial relationships has already been addressed in the literature and we just mention (in section 2) the approaches on which we base our work. We address the problem of inference in a probabilistic framework. We present the problem of inference of spatial relationships in the case of points (Section 3). We express the problem as the evaluation of the probability distribution of a point C knowing its relative position to another point B and the relative position of the point B to another fixed point A. This is an original formulation of the problem to the best of our knowledge. Inference is presented in Section 4.

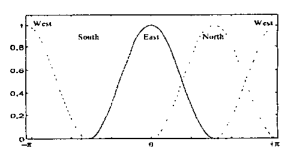
2. REPRESENTATION OF SPATIAL RELATIONSHIPS

Many of the existing methods for defining relative spatial position depend on angle measurements between points of the two objects of interest. Freeman [6] was among the firsts to recognize that the nature of spatial relations between objects requires that they be described in an approximate (fuzzy) framework. Here we give briefly the principle of some fuzzy approaches that are the basis of our work.

2.1. Directional spatial relationships definition between points



(a) Definition of θ_{AB}



(b) Fuzzy subsets which represent 4 basic spatial relations

Fig. 1. Evaluation of spatial relations between two points [1].

In [2, 1], the spatial relations between two points A and B are determined by the angle θ_{AB} (Fig. 1-a) made by the segment AB and the x-axis. A relative position relationship is then defined as a fuzzy set depending on θ. Fig. 1-b illustrates the four directional relations which are defined as trigonometric functions [1] of θ ($\cos^2\theta$ and $\sin^2\theta$). Other functions are possible: [2] uses trapeze-shaped functions. The membership function for a directional spatial relationship "rel" will be denoted by μ_{rel}, it is a function from $[-\pi, \pi[$ into $[0, 1]$. In the case of points, if rel is carrefuly chosen, the four fuzzy spatial relationships "East", "North", "West" and

"South" determine completely the angle between the segment joining the two points and the x-axis. Therefore, we can either use the angle θ_{AB} or the vector (East(AB), North(AB), West(AB), South $(AB))^t$ to represent the four cardinal relationships of the plane.

2.2. Directional spatial relationships definition between objects

In order to extend previous definitions to objects, a first simple solution consists in representing the two objects of interest by a particular point. In [7, 2], the two objects are represented by their centroids. The fuzzy spatial relationship between these two points defines the spatial relations between the two objects.

Miyajima and Ralescu [1] proposed the use of an angle histogram to represent directional spatial relationship between objects. The angle histogram H_{AB} is a function of θ. $H_{AB}(\theta)$ represents the frequency of occurrence of θ between any two points in objects A and B. To evaluate a specific directional spatial relationship "rel", a compatibility measure between H_{AB} and μ_{rel} is defined using compatibility [1] or fuzzy pattern matching [3]. Other approaches have also been proposed in the literature, but are not detailed here.

3. SPATIAL PROBABILITY OF C FROM RELATIVE POSITION θ_{AB} AND θ_{BC}

In this work, we study the problem of inference in a probabilistic framework. Since several approaches for defining spatial relationships between objects rely on the evaluation of spatial relations between points, we start with the evaluation of the probability distribution of a point C in the 2D plane limited to a circle knowing its relative spatial position with respect to a point B and the relative spatial position of B to another fixed point A.

3.1. Notation and hypotheses

We make use of polar coordinates and denote by (r_X, Θ_X) the coordinates of point X. For this study, we assume that:

1. All points are independent and distributed according to a uniform distribution in a disk of center O and radius R (i.e. the image is circular):

$$P(R \in [r - dr/2, r + dr/2], \Theta \in [\theta - d\theta/2,$$
$$\theta + d\theta/2])rdrd\theta = \begin{cases} \frac{1}{\pi R^2} rdrd\theta & \text{if } 0 \leq r \leq R \\ 0 & \text{else.} \end{cases} \quad (1)$$

2. We exclude degenerate cases where $\beta = \alpha + k\pi$.
3. The point A is located at the central point of the plane.
4. We assume that all angles are defined in $]-\pi, \pi]$.

We denote by $[x \pm y]$ the domain $[x - y, x + y]$, and by $\mathcal{B}$ the event "Point B belongs to the interval $[r_B \pm dr/2]$ in the direction $[\theta_B \pm d\theta/2]$":

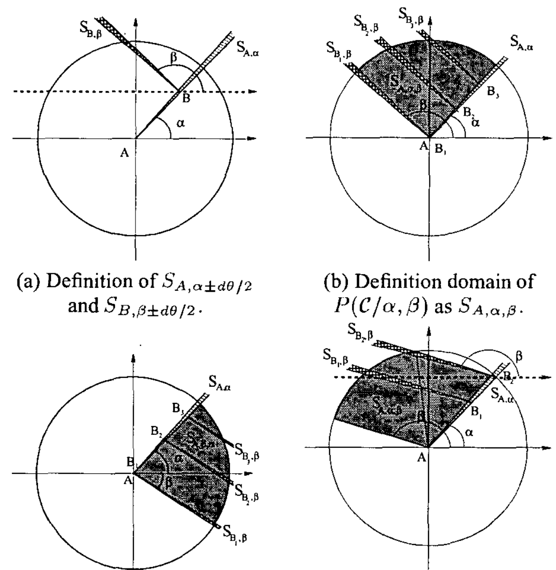
$$\mathcal{B} \equiv (R_B \in [r_B \pm dr/2] \wedge \Theta_B \in [\theta_B \pm d\theta/2])$$

In the same way:

$$\begin{aligned} \mathcal{C} &\equiv (R_C \in [r_C \pm dr/2] \wedge \Theta_C \in [\theta_C \pm d\theta/2]) \\ \alpha &\equiv (\theta_{AB} \in [\alpha \pm d\theta/2]) \\ \beta &\equiv (\theta_{BC} \in [\beta \pm d\theta/2]) \\ \gamma &\equiv (\theta_{AC} \in [\gamma \pm d\theta/2]) \end{aligned}$$

We have to evaluate the probability that "the point C belongs to the interval $[r_C \pm dr/2]$ in direction $[\theta_C \pm d\theta/2]$) knowing its relative spatial position with respect to another point B ($\theta_{BC} \in [\beta \pm d\theta/2]$) and knowing the relative spatial position of this point to the fixed point A ($\theta_{AB} \in [\alpha \pm d\theta/2]$)", irrespectively of r_B, i.e. to evaluate:

$$P(\mathcal{C}/\alpha, \beta)$$



(a) Definition of $S_{A,\alpha\pm d\theta/2}$ and $S_{B,\beta\pm d\theta/2}$.

(b) Definition domain of $P(\mathcal{C}/\alpha,\beta)$ as $S_{A,\alpha,\beta}$.

(c) Definition domain of $P(\mathcal{C}/\alpha,\beta)$ as $S_{A,\beta,\alpha}$.

(d) The case where the definition domain is an incomplete angular sector.

Fig. 2. Definition domains of $P(\mathcal{C}/\alpha,\beta)$.

3.2. Definition domains of $P(\mathcal{C}/\alpha, \beta)$

The point B being in the direction α with respect to point A, it belongs to the angular sector $S_{A,\alpha\pm d\theta/2}$ of origin A (Fig. 2-a). As point C is in the direction β w.r.t B, C belongs to the angular sector $S_{B,\beta\pm d\theta/2}$ of origin B (Fig. 2-a). When B runs for all points of the angular sector $S_{A,\alpha\pm d\theta/2}$, the sector $S_{B,\beta\pm d\theta/2}$ describes an angular sector $S_{A,\alpha,\beta}$ (Fig. 2-b). Therefore, if $C \notin S_{A,\alpha,\beta}$, $P(\mathcal{C}/\alpha,\beta) = 0$. This sector is as in Fig. 2-b if $(\beta - \alpha) \in [2k\pi, \pi + 2k\pi]$, and as in Fig. 2-c (i.e. $S_{A,\beta,\alpha}$) when $(\beta - \alpha) \in [\pi + 2k\pi, 2\pi + 2k\pi]$.

For each case, the defined sector is not a complete angular sector if $(\beta - \alpha) \in [\pi/2 + 2k\pi, \pi + 2k\pi]$ (for the first case Fig. 2-d) or $(\beta - \alpha) \in [\pi + 2k\pi, 3\pi/2 + 2k\pi]$ (for the second case), because there is a part of the angular sector for which the corresponding points B do not belong the defined disk. So we have four cases, the two first ones being symmetrical of the two others. We present the results for the two first cases:

- $(\beta - \alpha) \in [2k\pi, \pi/2 + 2k\pi]$: the defined sector is a complete angular sector $S_{A,\alpha,\beta}$ (Fig. 2-b),
- $(\beta - \alpha) \in [\pi/2 + 2k\pi, \pi + 2k\pi]$: the defined sector is an incomplete angular sector $S_{A,\alpha,\beta}$ (Fig. 2-d).

3.3. Evaluation of $P(\mathcal{C}/\alpha, \beta)$

By introducing the position of B (event $\mathcal{B}$), we have:

$$P(\mathcal{C}/\alpha, \beta) = \frac{\int_{\mathcal{D}} P(\mathcal{C}, \alpha, \beta, \mathcal{B}) d\mathcal{B}}{P(\alpha, \beta)} \quad (2)$$

$$\int_{\mathcal{D}} P(\mathcal{C}, \alpha, \beta, \mathcal{B}) d\mathcal{B} = \int_{\mathcal{D}} P(\beta/\mathcal{B}, \mathcal{C}, \alpha) P(\alpha/\mathcal{B}, \mathcal{C}) P(\mathcal{B}) P(\mathcal{C}) d\mathcal{B} \quad (3)$$

The angle α between the segment joining the two points A and B and the x-axis is independent on the position of the point C, so:

$$P(\alpha/\mathcal{B}, \mathcal{C}) = P(\alpha/\mathcal{B}) = \begin{cases} 1 & \text{if } B \in S_{A,\alpha \pm d\theta} \\ 0 & else \end{cases} \quad (4)$$

From the uniform distribution we deduce:

$$P(\mathcal{B}) = P(\mathcal{C}) = \frac{1}{\pi R^2} \quad (5)$$

We denote $K = \frac{1}{\pi R^2}$.

$$\int_{\mathcal{D}} P(\mathcal{C}, \alpha, \beta, \mathcal{B}) d\mathcal{B} = K^2 \int_{S_{A,\alpha \pm d\theta}} P(\beta/\mathcal{B}, \mathcal{C}, \alpha) d\mathcal{B} \quad (6)$$

The sentence "C is in the direction β with respect to B" is written as:

$$P(\beta/\mathcal{B}, \mathcal{C}, \alpha) = \begin{cases} 1 & \text{if } C \in S_{B,\beta \pm d\theta} \\ 0 & \text{else} \end{cases} \quad (7)$$

or similarly as:

$$P(\beta/\mathcal{B}, \mathcal{C}, \alpha) = \begin{cases} 1 & \text{if } B \in S_{C,(\beta-\pi) \pm d\theta} \\ 0 & \text{else} \end{cases} \quad (8)$$

Using Eq. 6 and Eq. 8 we obtain:

$$\int_{\mathcal{D}} P(\mathcal{C}, \alpha, \beta, \mathcal{B}) d\mathcal{B} = K^2 \int_{\rho} d\mathcal{B} \quad (9)$$

with $\rho = S_{A,\alpha \pm d\theta} \cap S_{C,(\beta-\pi) \pm d\theta}$. So the integral (Eq. 9) is proportional to the surface s of ρ which is given by :

$$s = 2 r_c^2 \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta - 2\gamma)}{3 \sin(\alpha - \beta) - \sin(3\alpha - 3\beta)} \quad (10)$$

The probability $P(\alpha, \beta)$ is obtained by:

$$P(\alpha, \beta) = \int_{\mathcal{D}} \int_{\mathcal{D}} P(\mathcal{C}, \alpha, \beta, \mathcal{B}) d\mathcal{B}\, d\mathcal{C} \quad (11)$$

We use (Eq. 9) to evaluate $P(\alpha, \beta)$ when $\mathcal{C}$ belongs to the angular sector $S_{A,\alpha,\beta}$. For the complete angular sector:

$$P(\alpha, \beta) = \frac{(\alpha - \beta) \cos(\alpha - \beta) - \sin(\alpha - \beta)}{2\pi^2 (\sin(3\alpha - 3\beta) - 3 \sin(\alpha - \beta))} \quad (12)$$

and a rather more complex expression in the other case [8]:

$$P(\alpha, \beta) = \frac{5 \sin(\alpha - \beta) + \sin(3\alpha - 3\beta) - 3 \sin(5\alpha - 5\beta) + \sin(7\alpha - 7\beta)}{8\pi^2 (10 \sin(\alpha - \beta) - 5 \sin(3\alpha - 3\beta) + \sin(5\alpha - 5\beta))} + \frac{2(\beta - \alpha - \pi) \cos(\alpha - \beta) - \sin(\alpha - \beta) + \sin(3\alpha - 3\beta)}{4\pi^2 (\sin(3\alpha - 3\beta) - 3 \sin(\alpha - \beta))} \quad (13)$$

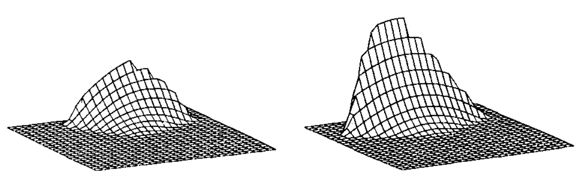
To keep a simple notation for the two cases, we denote $K_1 = \frac{2K^2}{P(\alpha,\beta)}$ which is not the same for the two cases:

$$P(\mathcal{C}/\alpha, \beta) = \begin{cases} K_1 r_c^2 \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta) - \sin(3\alpha-3\beta)} & \text{if } C \in S_{A,\alpha,\beta} \\ 0 & \text{else} \end{cases} \quad (14)$$

which is the wanted distribution of probability of C.

3.4. Discussion

Equation 14 allows to compute the position probability of a point C which is known to be in the direction β of a point B, itself in the direction α from the central point of the image subject to a uniform and independent distribution of B and C. Not surprisingly, we see from Fig. 3 that the probability is non-zero only in the region limited by angles α and β. This region is complete (we say complete sector) iff $(\beta - \alpha) \in [2k\pi, \pi/2 + 2k\pi]$, it is an incomplete angular sector in the other case. In the case of complete sector the probability peaks at its maximum value for the angle $\Theta_C = \frac{\alpha+\beta}{2}$, and on the border of the disk ($R_C = R$) which is, by the way, the most probable point C. In the case of incomplete sector, we distinguish two cases, if $\frac{\alpha+\beta}{2} > (2\beta - \alpha - \pi)$ (i.e. $\beta - \alpha < \frac{2\pi}{3}$), the intersection between the bisector and the border of the disk belongs to the domain $S_{A,\alpha,\beta}$ and the probability peaks at its maximum value for the point $C = (R, \frac{\alpha+\beta}{2})$. In the other case, the maximum is given by the point C located at the intersection between the border of the disk and the border of the triangular part of the incomplete angular sector ($C = (R, 2\beta - \alpha - \pi)$).



(a) $P(\mathcal{C}/\alpha = \pi/6, \beta = 2\pi/3)$. (b) $P(\mathcal{C}/\alpha = \pi/6, \beta = 5\pi/6)$.

Fig. 3. Examples of the probability distribution $P(\mathcal{C}/\alpha, \beta)$.

4. INFERRING SPATIAL RELATIONSHIPS

The basic relationship we are using is $\text{rel}(X, Y) = \theta_{XY}$, expressing that Y is in the direction θ_{XY} with respect to X. From $\text{rel}(A, B) = \alpha$ and $\text{rel}(B, C) = \beta$, we want to deduce $\text{rel}(A, C)$.

A first step is to determine $P(\gamma/\alpha,\beta)$ which amounts to compute the integral of $P(C/\alpha,\beta)$ over the angular sector S of angle $\gamma \pm d\gamma/2$ to eliminate the dependence in r_c:

$$P(\gamma/\alpha,\beta) = \int_{S_{A,\gamma\pm d\gamma/2}} P(C/\alpha,\beta)dC \qquad (15)$$

To evaluate $P(\gamma/\alpha,\beta)$ we distinguish two cases:
case 1: $(\beta-\alpha) \in [2k\pi, \pi/2+2k\pi]$: $S_{A,\alpha,\beta}$ is a complete angular sector.

$$P(\gamma/\alpha,\beta) = \begin{cases} K_1 \frac{R^4}{4} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} & \text{if } \gamma \in [\alpha+2k\pi, \beta+2k\pi] \\ 0 & \text{else} \end{cases}$$

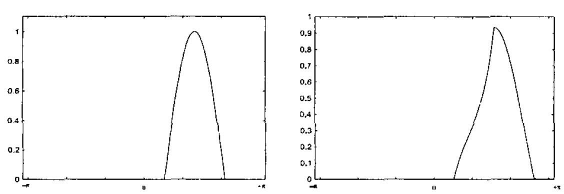
case 2: $(\beta-\alpha) \in [\pi/2+2k\pi, \pi+2k\pi]$: $S_{A,\alpha,\beta}$ is an incomplete angular sector. In this case, the triangular part is defined by the angle $\gamma \in [\alpha+2k\pi, (2\beta-\alpha-\pi)+2k\pi]$. In this case the radius of the angular sector $S_{A,\gamma\pm d\theta}$ is:

$$\begin{cases} R\frac{\sin(\beta-\alpha)}{\sin(\beta-\gamma)} & \text{if } \gamma \in [\alpha+2k\pi, (2\beta-\alpha-\pi)+2k\pi] \\ R & \text{else} \end{cases}$$

So:

$$P(\gamma/\alpha,\beta) = \begin{cases} K_1 \frac{\left(R\frac{\sin(\beta-\alpha)}{\sin(\beta-\gamma)}\right)^4}{4} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} \\ \text{if } \gamma \in [\alpha+2k\pi, (2\beta-\alpha-\pi)+2k\pi] \\ K_1 \frac{R^4}{4} \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)}{3\sin(\alpha-\beta)-\sin(3\alpha-3\beta)} \\ \text{if } \gamma \in [(2\beta-\alpha-\pi)+2k\pi, \beta+2k\pi] \\ 0 & \text{else} \end{cases} \qquad (16)$$

From $P(\gamma/\alpha,\beta)$ several solutions exist to estimate rel(A,C). We



(a) $P(\gamma/\alpha=\pi/6, \beta=2\pi/3)$. (b) $P(\gamma/\alpha=\pi/6, \beta=5\pi/6)$.

Fig. 4. Examples of the distribution probability $P(\gamma/\alpha,\beta)$ for a complete sector (left) and an incomplete one (right).

can use the a maximum posterior probability estimator:

$$\text{rel}_1(A,C) = \arg\sup_{\gamma} P(\gamma/\alpha,\beta) \qquad (17)$$

We can also use the maximum likelihood estimator:

$$\text{rel}_2(A,C) = \arg\sup_{\gamma} P(\alpha,\beta/\gamma) \qquad (18)$$

These two estimators are linked by Bayes' rule:

$$\text{rel}_2(A,C) = \arg\sup_{\gamma} \frac{P(\gamma/\alpha,\beta)P(\alpha,\beta)}{P(\gamma)} \qquad (19)$$

The probability $P(\alpha,\beta)$ is independent on the variable γ, so:

$$\text{rel}_2(A,C) = \arg\sup_{\gamma} \frac{P(\gamma/\alpha,\beta)}{P(\gamma)} \qquad (20)$$

If we assume that all angles γ are equiprobable events ($P(\gamma) = Cst$), both results are the same. But for some problems we may have different hypotheses for $P(\gamma)$ which will provide different estimates in the MAP case and maximum likelihood case.

In the case of complete angular sector, the probability $P(\gamma/\alpha,\beta)$ peaks at its maximum value for the angle $\Gamma = \frac{\alpha+\beta}{2}$. In the other case, we distinguish two cases, if $(\beta-\alpha) < \frac{2\pi}{3}$, the maximum is given by the angle $\Gamma = \frac{\alpha+\beta}{2}$. In the other case the maximum is given by the angle corresponding to the intersection between the border of the disk and the border of the incomplete angular sector $\Gamma = 2\beta - \alpha - \pi$.

All the results which are derived here in the case of circular windows have also been extended to the more complex case of rectangular images [8].

5. CONCLUSION

We proposed in this paper an original formulation of the problem of inference of directional spatial relationships in the case of points. We have used a probabilistic framework to represent our problem. The application of these results in real applications requires to extend this work to objects. We are currently working on this extension, to infer spatial relations between objects (and not only between points), by using angle histogram representation as in 2.2.

6. REFERENCES

[1] K. Miyajima and A. Ralescu, "Spatial organisation in 2D segmented images: Representation and recognition of primitive spatial relations," *Fuzzy Sets and Systems*, vol. 65, pp. 225–236, 1994.

[2] J.M. Keller and X. Wang, "Comparison of spatial relation definitions in computer vision," in *IEEE Procedings of ISUMA-NAFIPS*, Sept. 1995, pp. 679–684.

[3] I. Bloch, "Fuzzy relative position between objects in images: a morphological approach," in *IEEE Int. Conf. on Image Processing ICIP'96*, Lausanne, Sept. 1996, vol. II, pp. 987–990.

[4] I. Bloch, "Fuzzy relative position between objects in image processing: a morphological approach," *IEEE Trans. on PAMI*, vol. 21, no. 5, pp. 657–664, July 1999.

[5] S. Dutta, "Approximate spatial reasoning: integrating qualitative and quantitative constraints," *International Journal of Approximate Reasoning*, vol. 5, pp. 307–331, 1991.

[6] J. Freeman, "The modeling of spatial relations," *Computer Graphics and Image Processing*, vol. 4, pp. 156–171, 1975.

[7] R. Krishnapuram, J.M. Keller, and Y. Ma, "Quantitative analysis of properties and spatial relations of fuzzy image regions," *IEEE Trans. on Fuzzy Systems*, vol. 1, no. 3, pp. 222–233, Aug. 1993.

[8] S.M.R. Dehak, "Inference of directional spatial relationship between points: A probabilistic approach," Tech. Rep., ENST, Paris, 2001.