Bipolar Fuzzy Mathematical Morphology for Spatial Reasoning

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Abstract. Bipolarity is an important feature of spatial information, involved in the expressions of preferences and constraints about spatial positioning, or in pairs of "opposite" spatial relations such as left and right. Imprecision should also be taken into account, and fuzzy sets is then an appropriate formalism. In this paper, we propose to handle such information based on mathematical morphology operators, extended to the case of bipolar fuzzy sets. The potential of this formalism for spatial reasoning is illustrated on a simple example in brain imaging.

Keywords: bipolar spatial information, fuzzy sets, spatial relations, bipolar fuzzy dilation and erosion, spatial reasoning.

1 Introduction

Spatial reasoning includes two main aspects: knowledge representation, concerning spatial entities and spatial relations, and reasoning on them. In this paper, we consider both imprecision and bipolarity of spatial information. Imprecision should be taken into account to represent vague knowledge about spatial positions or spatial relations (typically directional relations such as left and right) [1]. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [2]. The intersection of the positive information and the negative information has to be empty in order to achieve consistency of the representation, and their union does not necessarily cover the whole underlying space, i.e. there is no direct duality between both types of information, leaving room for indifference or indetermination. In this paper, we consider bipolarity of spatial information and propose to handle it as bipolar fuzzy sets (Section 2) using mathematical morphology operators, extended to these representations (Section 3). Some additional properties are included with respect to our previous work [3,4]. We then present some examples of spatial reasoning in Section 4, as the main contribution of this paper.

2 Bipolar Fuzzy Sets

Let $\mathcal{S}$ be the underlying space (the spatial domain for spatial information processing), that is supposed to be bounded and finite here. A bipolar fuzzy set on

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$\mathcal{S}$ is defined by a pair of functions (μ, ν) such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ the non-membership degree (negative information), while $1 - \mu(x) - \nu(x)$ encodes a degree of neutrality, indifference or indetermination. This formalism allows representing both bipolarity and fuzziness. Concerning semantics, it should be noted that a bipolar fuzzy set does not necessarily represent one physical object or spatial entity, but rather more complex information, potentially issued from different sources.

Let us consider the set $\mathcal{L}$ of pairs of numbers (a, b) in $[0, 1]$ such that $a + b \leq 1$. It is a complete lattice, for the partial order defined as [5]: $(a_1, b_1) \preceq (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \geq b_2$. The greatest element is $(1, 0)$ and the smallest element is $(0, 1)$. The supremum and infimum are respectively defined as: $(a_1, b_1) \vee (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2))$, $(a_1, b_1) \wedge (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2))$. The partial order $\preceq$ induces a partial order on the set of bipolar fuzzy sets:

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \text{ iff } \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x) \text{ and } \nu_1(x) \geq \nu_2(x), \quad (1)$$

and infimum and supremum are defined accordingly. It follows that, if $\mathcal{B}$ denotes the set of bipolar fuzzy sets on $\mathcal{S}$, $(\mathcal{B}, \preceq)$ is a complete lattice.

3 Bipolar Fuzzy Erosion and Dilation

Mathematical morphology on bipolar fuzzy sets has been first introduced in [3]. Once we have a complete lattice, as described in Section 2, it is easy to define algebraic dilations and erosions on this lattice, as operators that commute with the supremum and the infimum, respectively [3]. Their properties are derived from general properties of lattice operators. If we assume that $\mathcal{S}$ is an affine space (or at least a space on which translations can be defined), it is interesting, for dealing with spatial information, to consider morphological operations based on a structuring element. We detail the construction of such morphological operators, extending our preliminary work in [3,4].

Erosion. As for fuzzy sets [6], defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets [7], which are formally (although not semantically) equivalent to bipolar fuzzy sets. With our notations, a degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as:

$$\inf_{x \in \mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \quad (2)$$

where I is an implication operator. Two types of implication can be defined [7], one derived from a bipolar t-conorm $\perp$[1]:

[1] A bipolar disjunction is an operator D from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $D((1, 0), (1, 0)) = D((0, 1), (1, 0)) = D((1, 0), (0, 1)) = (1, 0)$, $D((0, 1), (0, 1)) = (0, 1)$ and that is increasing in both arguments. A bipolar t-conorm is a commutative and associative bipolar disjunction such that the smallest element of $\mathcal{L}$ is the unit element.

$$I_N((a_1, b_1), (a_2, b_2)) = \bot((b_1, a_1), (a_2, b_2)), \tag{3}$$

and one derived from a residuation principle from a bipolar t-norm $\top$[2]:

$$I_R((a_1, b_1), (a_2, b_2)) = \sup\{(a_3, b_3) \in \mathcal{L} \mid \top((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2)\} \tag{4}$$

where $(a_i, b_i) \in \mathcal{L}$ and (b_i, a_i) is the standard negation of (a_i, b_i).

Two types of t-norms and t-conorms are considered in [7] and will be considered here as well:

1. operators called t-representable t-norms and t-conorms, which can be expressed using usual t-norms t and t-conorms T from the fuzzy sets theory [8]:

$$\top((a_1, b_1), (a_2, b_2)) = (t(a_1, a_2), T(b_1, b_2)), \tag{5}$$

$$\bot((a_1, b_1), (a_2, b_2)) = (T(a_1, a_2), t(b_1, b_2)). \tag{6}$$

2. Lukasiewicz operators, which are not t-representable:

$$\top_W((a_1, b_1), (a_2, b_2)) = (\max(0, a_1 + a_2 - 1), \min(1, b_1 + 1 - a_2, b_2 + 1 - a_1)), \tag{7}$$

$$\bot_W((a_1, b_1), (a_2, b_2)) = (\min(1, a_1 + 1 - b_2, a_2 + 1 - b_1), \max(0, b_1 + b_2 - 1)). \tag{8}$$

In these equations, the positive part of $\top_W$ is the usual Lukasiewicz t-norm of a_1 and a_2 (i.e. the positive parts of the input bipolar values). The negative part of $\bot_W$ is the usual Lukasiewicz t-norm of the negative parts (b_1 and b_2) of the input values. The two types of implication coincide for the Lukasiewicz operators [5].

Based on these concepts, we can now propose a definition for morphological erosion.

Definition 1. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The erosion of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from an implication I as:*

$$\forall x \in \mathcal{S}, \varepsilon_{(\mu_B, \nu_B)}((\mu, \nu))(x) = \inf_{y \in \mathcal{S}} I((\mu_B(y - x), \nu_B(y - x)), (\mu(y), \nu(y))), \tag{9}$$

where $\mu_B(y - x)$ denotes the value at point y of μ_B translated at x.

A similar approach has been used for intuitionistic fuzzy sets in [9], but with weaker properties (in particular an important property such as the commutativity of erosion with the conjunction may be lost).

[2] A bipolar conjunction is an operator C from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $C((0, 1), (0, 1)) = C((0, 1), (1, 0)) = C((1, 0), (0, 1)) = (0, 1)$, $C((1, 0), (1, 0)) = (1, 0)$ and that is increasing in both arguments. A bipolar t-norm is a commutative and associative bipolar conjunction such that the largest element of $\mathcal{L}$ is the unit element.

Morphological dilation of bipolar fuzzy sets. Dilation can be defined based on a duality principle or based on the adjunction property. Both approaches have been developed in the case of fuzzy sets, and the links between them and the conditions for their equivalence have been proved in [10,11]. Similarly we consider both approaches to define morphological dilation on $\mathcal{B}$.

Dilation by duality. The duality principle states that the dilation is equal to the complementation of the erosion, by the same structuring element (if it is symmetrical with respect to the origin of $\mathcal{S}$, otherwise its symmetrical is used), applied to the complementation of the original set. Applying this principle to bipolar fuzzy sets using a complementation c (typically the standard negation $c((a,b)) = (b,a)$) leads to the following definition of morphological bipolar dilation.

Definition 2. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element. The dilation of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from erosion by duality as:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu)) = c[\varepsilon_{(\mu_B,\nu_B)}(c((\mu,\nu)))]. \tag{10}$$

Dilation by adjunction. Let us now consider the adjunction principle, as in the general algebraic case. An adjunction property can also be expressed between a bipolar t-norm and the corresponding residual implication as follows:

$$\top((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2) \Leftrightarrow (a_3,b_3) \preceq I_R((a_1,b_1),(a_2,b_2)). \tag{11}$$

Definition 3. *Using a residual implication for the erosion for a bipolar t-norm $\top$, the bipolar fuzzy dilation, adjoint of the erosion, is defined as:*

$$\begin{aligned} \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \inf\{(\mu',\nu')(x) \mid (\mu,\nu)(x) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu'))(x)\} \\ &= \sup_{y\in\mathcal{S}} \top((\mu_B(x-y),\nu_B(x-y)),(\mu(y),\nu(y))). \end{aligned} \tag{12}$$

Links between both approaches. It is easy to show that the bipolar Lukasiewicz operators are adjoint, according to Equation 11. It has been shown that the adjoint operators are all derived from the Lukasiewicz operators, using a continuous bijective permutation on $[0,1]$ [7]. Hence equivalence between both approaches can be achieved only for this class of operators. This result is similar to the one obtained for fuzzy mathematical morphology [10,11].

An illustrative example is shown in Figure 1.

Properties.

Proposition 1. *All definitions are consistent: they actually provide bipolar fuzzy sets of $\mathcal{B}$.*

Proposition 2. *In case the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1-\mu$ and $\nu_B = 1-\mu_B$), the definitions lead to the usual definitions of fuzzy dilations and erosions (using classical Lukasiewicz t-norm and t-conorm for the definitions based on the Lukasiewicz operators). Hence they are also compatible with classical morphology in case μ and μ_B are crisp.*

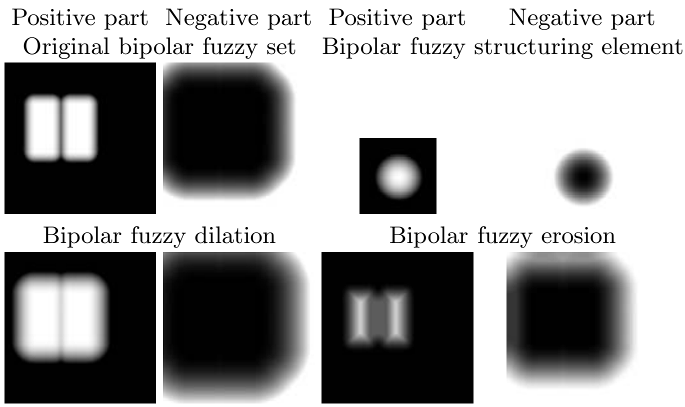

Fig. 1. Bipolar fuzzy set and structuring element, dilation and erosion

Proposition 3. *The proposed definitions of bipolar fuzzy dilations and erosions commute respectively with the supremum and the infinum of the lattice* $(\mathcal{B}, \preceq)$.

Proposition 4. *The bipolar fuzzy dilation is extensive (i.e.* $(\mu,\nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu))$*) and the bipolar fuzzy erosion is anti-extensive (i.e.* $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu,\nu)$*) if and only if* $(\mu_B,\nu_B)(0) = (1,0)$*, where* 0 *is the origin of the space* $\mathcal{S}$ *(i.e. the origin completely belongs to the structuring element, without any indetermination).*

Note that this condition is equivalent to the conditions on the structuring element found in classical and fuzzy morphology to have extensive dilations and anti-extensive erosions [12,6].

Proposition 5. *The dilation satisfies the following iterativity property:*

$$\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}((\mu,\nu))) = \delta_{(\delta_{\mu_B}(\mu'_B),1-\delta_{(1-\nu_B)}(1-\nu'_B))}((\mu,\nu)). \quad (13)$$

Proposition 6. *Conversely, if we want all classical properties of mathematical morphology to hold true, the bipolar conjunctions and disjunctions used to define intersection and inclusion in* $\mathcal{B}$ *have be be bipolar t-norms and t-conorms. If both duality and adjunction are required, then the only choice is bipolar Lukasiewicz operators (up to a continuous permutation on* $[0,1]$*).*

This new result is very important, since it shows that the proposed definitions are the most general ones to have a satisfactory interpretation in terms of mathematical morphology.

Interpretations. Let us first consider the implication defined from a t-representable bipolar t-conorm. Then the erosion is written as:

$$\begin{aligned}\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \inf_{y\in\mathcal{S}} \bot((\nu_B(y-x),\mu_B(y-x)),(\mu(y),\nu(y))) \\ &= (\inf_{y\in\mathcal{S}} T((\nu_B(y-x),\mu(y)), \sup_{y\in\mathcal{S}} t(\mu_B(y-x),\nu(y))). \quad (14)\end{aligned}$$

This resulting bipolar fuzzy set has a membership function which is exactly the fuzzy erosion of μ by the fuzzy structuring element $1-\nu_B$, according to the original definitions in the fuzzy case [6]. The non-membership function is exactly the dilation of the fuzzy set ν by the fuzzy structuring element μ_B.

Let us now consider the derived dilation, based on the duality principle. Using the standard negation, it is written as:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = (\sup_{y\in\mathcal{S}} t(\mu_B(x-y),\mu(y)), \inf_{y\in\mathcal{S}} T((\nu_B(x-y),\nu(y))). \quad (15)$$

The first term (membership function) is exactly the fuzzy dilation of μ by μ_B, while the second one (non-membership function) is the fuzzy erosion of ν by $1-\nu_B$, according to the original definitions in the fuzzy case [6].

This observation has a nice interpretation, which well fits with intuition. Let (μ,ν) represent a spatial bipolar fuzzy set, where μ is a positive information for the location of an object for instance, and ν a negative information for this location. A bipolar structuring element can represent additional imprecision on the location, or additional possible locations. Dilating (μ,ν) by this bipolar structuring element amounts to dilate μ by μ_B, i.e. the positive region is extended by an amount represented by the positive information encoded in the structuring element. On the contrary, the negative information is eroded by the complement of the negative information encoded in the structuring element. This corresponds well to what would be intuitively expected in such situations. A similar interpretation can be provided for the bipolar fuzzy erosion.

Similarly, if we now consider the implication derived from the Lukasiewicz bipolar operators (Equations 7 and 8), it is easy to show that the negative part of the erosion is exactly the fuzzy dilation of ν (negative part of the input bipolar fuzzy set) with the structuring element μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Similarly, the positive part of the dilation is the fuzzy dilation of μ (positive part of the input) by μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Hence for both operators, the "dilation" part (i.e. negative part for the erosion and positive part for the dilation) has always a direct interpretation and is the same as the one obtained using t-representable operators, for t being the Lukasiewicz t-norm.

In the case the structuring element is non bipolar (i.e. $\forall x \in \mathcal{S}, \nu_B(x) = 1 - \mu_B(x)$), then the "erosion" part has also a direct interpretation: the positive part of the erosion is the fuzzy erosion of μ by μ_B for the Lukasiewicz t-conorm; the negative part of the dilation is the erosion of ν by μ_B for the Lukasiewicz t-conorm.

4 Application to Spatial Reasoning

Mathematical morphology provides tools for spatial reasoning at several levels [13]. Its features allow representing objects or object properties, that we do not address here to concentrate rather on tools for representing spatial relations. The notion of structuring element captures the local spatial context, in a fuzzy and bipolar way here, which endows dilation and erosion with a low level spatial reasoning feature, as shown in the interpretation part of Section 3. This is then reinforced by the derived operators (opening, closing, gradient, conditional operations...), as introduced for bipolar fuzzy sets in [14]. At a more global level, several spatial relations between spatial entities can be expressed as morphological operations, in particular using dilations [1,13], leading to large scale spatial reasoning, based for instance on distances [15].

Let us provide a few examples where bipolarity occurs when dealing with spatial information, in image processing or for spatial reasoning applications: when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indifference or neutrality. This corresponds to the idea that the union of positive and negative information does not cover all the space. Similar considerations can be provided for other pairs of "opposite" relations, such as "close to" and "far from" for instance.

In this section, we illustrate a typical scenario showing the interest of bipolar representations of spatial relations and of morphological operations on these representations for spatial reasoning.

An example of a brain image is shown in Figure 2, with a few labeled structures of interest.

Let us first consider the right hemisphere (i.e. the non-pathological one). We consider the problem of defining a region of interest for the RPU, based on a known segmentation of RLV and RTH. An anatomical knowledge base or

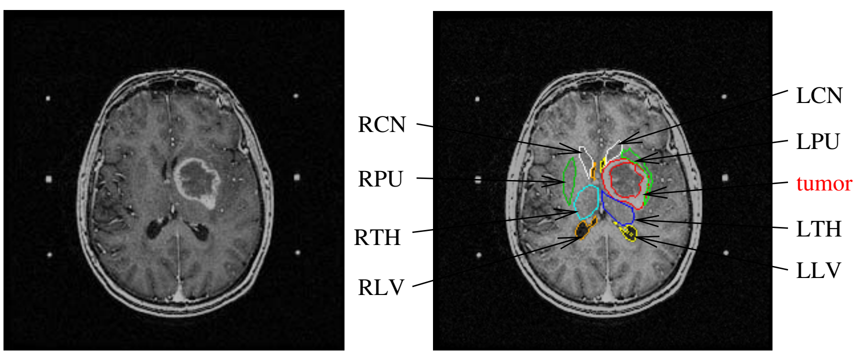

Fig. 2. A slice of a 3D MRI brain image, with a few structures: left and right lateral ventricles (LLV and RLV), caudate nuclei (LCN and RCN), putamen (LPU and RPU) and thalamus (LTH and RTH). A ring-shaped tumor is present in the left hemisphere (the usual "left is right" convention is adopted for the visualization).

ontology provides some information about the relative position of these structures [16,17]:

- directional information: the RPU is exterior (left on the image) of the union of RLV and RTH (positive information) and cannot be interior (negative information);
- distance information: the RPU is quite close to the union of RLV and RTH (positive information) and cannot be very far (negative information).

These pieces of information are represented in the image space based on morphological dilations using appropriate structuring elements [1] (representing the semantics of the relations, as displayed in Figure 3) and are illustrated in Figure 4. A bipolar fuzzy set modeling the direction information is defined as:

$$(\mu_{dir}, \nu_{dir}) = (\delta_{\nu_L}(\text{RLV} \cup \text{RTH}), \delta_{\nu_R}(\text{RLV} \cup \text{RTH})),$$

where ν_L and ν_R define the semantics of left and right, respectively. Similarly a bipolar fuzzy set modeling the distance information is defined as:

$$(\mu_{dist}, \nu_{dist}) = (\delta_{\nu_C}(\text{RLV} \cup \text{RTH}), 1 - \delta_{1-\nu_F}(\text{RLV} \cup \text{RTH})),$$

where ν_C and ν_F define the semantics of close and far, respectively. The neutral area between positive and negative information allows accounting for potential anatomical variability. The conjunctive fusion of the two types of relations is computed as a conjunction of the positive parts and a disjunction of the negative parts:

$$(\mu_{Fusion}, \nu_{Fusion}) = (\min(\mu_{dir}, \mu_{dist}), \max(\nu_{dir}, \nu_{dist})).$$

As shown in the illustrated example, the RPU is well included in the bipolar fuzzy region of interest which is obtained using this procedure. This region can then be efficiently used to drive a segmentation and recognition technique of the RPU.

Let us now consider the left hemisphere, where a ring-shaped tumor is present. The tumor induces a deformation effect which strongly changes the shape of the normal structures, but also their spatial relations, to a less extent. In particular the LPU is pushed away from the inter-hemispheric plane, and the LTH is pushed towards the posterior part of the brain and compressed. Applying the same procedure as for the right hemisphere does not lead to very satisfactory results in this case (see Figure 6). The default relations are here too strict and the resulting region of interest is not adequate: the LPU only satisfies with low

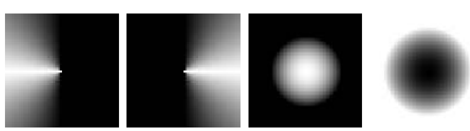

Fig. 3. Fuzzy structuring elements ν_L, ν_R, ν_C and ν_F, defining the semantics of left, right, close and far, respectively

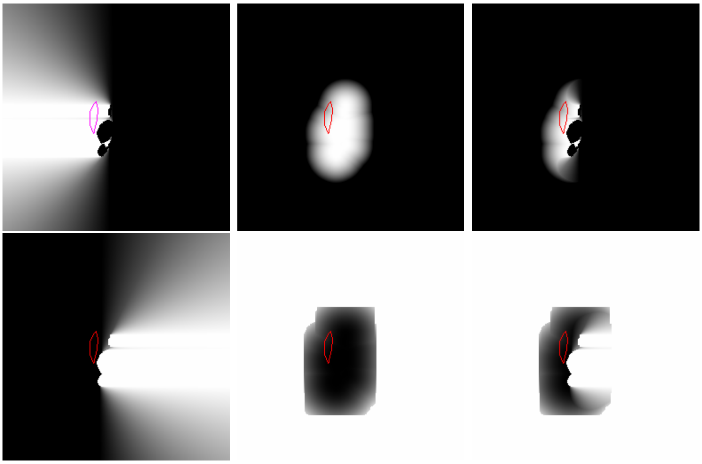

Fig. 4. Bipolar fuzzy representations of spatial relations with respect to RLV and RTH. Top: positive information, bottom: negative information. From left to right: directional relation, distance relation, conjunctive fusion. The contours of the RPU are displayed to show the position of this structure with respect to the region of interest.

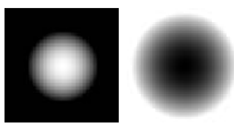

Fig. 5. Bipolar fuzzy structuring element (μ_{var}, ν_{var})

degrees the positive part of the information, while it also slightly overlaps the negative part. In such cases, some relations (in particular metric ones) should be considered with care. This means that they should be more permissive, so as to include a larger area in the possible region, accounting for the deformation induced by the tumor. This can be easily modeled by a bipolar fuzzy dilation of the region of interest with a structuring element (μ_{var}, ν_{var}) (Figure 5), as shown in the last column of Figure 6:

$$(\mu'_{dist}, \nu'_{dist}) = \delta_{(\mu_{var}, \nu_{var})}(\mu_{dist}, \nu_{dist}),$$

where (μ_{dist}, ν_{dist}) is defined as for the other hemisphere. Now the obtained region is larger but includes the correct area. This bipolar dilation amounts to dilate the positive part and to erode the negative part, as explained in Section 3.

Let us finally consider another example, where we want to use symmetry information to derive a search region for a structure in one hemisphere, based on the segmentation obtained in the other hemisphere. As an illustrative example, we consider the thalamus, and assume that it has been segmented in the non pathological hemisphere (right). Its symmetrical with respect to the

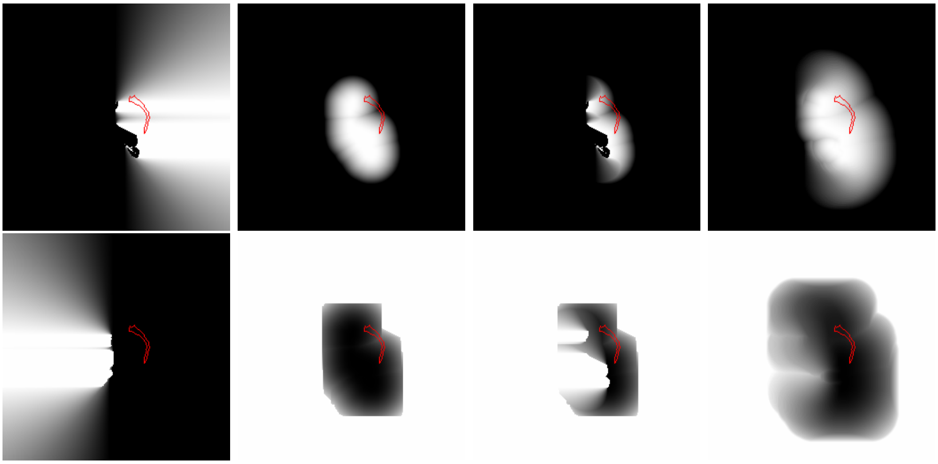

Fig. 6. Bipolar fuzzy representations of spatial relations with respect to LLV and LTH. From left to right: directional relation, distance relation, conjunctive fusion, Bipolar fuzzy dilation. First line: positive parts, second line: negative parts. The contours of the LPU are displayed to show the position of this structure.

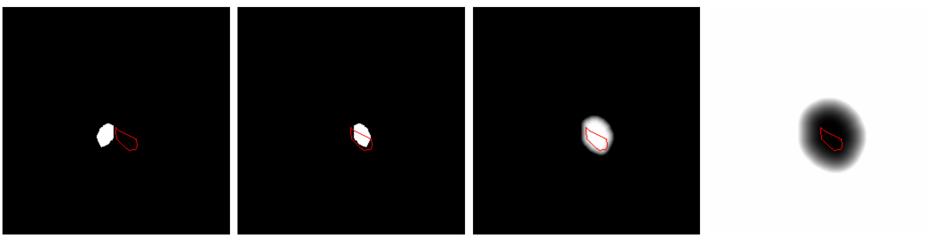

Fig. 7. RTH and its symmetrical, bipolar dilation defining an appropriate search region for the LTH (left: positive part, right: negative part)

inter-hemispheric plane should provide an adequate search region for the LTH in normal cases. Here this is not case, because of the deformation induced by the tumor (see Figure 7). Since the brain symmetry is approximate, a small deviation could be expected, but not as large as the one observed here. Here again a bipolar dilation allows defining a proper region, by taking into account both the deformation induced by the tumor and the imprecision in the symmetry.

5 Conclusion

In this paper, we have shown how a formal extension of mathematical morphology operators to the lattice of bipolar fuzzy sets may be used to represent two important features of spatial information, imprecision on the one hand and bipolarity on the other hand. This formalism can be useful for spatial reasoning, as illustrated on a typical scenario in brain imaging.

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