ON THE IMPLEMENTATION OF THE MULTI-PHASE REGION SEGMENTATION, SOLVING THE HIDDEN PHASE PROBLEM

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ABSTRACT

We consider the Chan and Vese multiphase segmentation based on the partition of an image minimizing an energy involving a region-based data fidelity term and a regularization term. The common implementation of the continuous optimization of this segmentation framework, with multiple level set functions, raises some numerical issues which lead to poor performance of the method when handling more than two phases. We propose a general formulation of the multi-phase model, and a permutation method, incorporated in the level-set based implementation of the multi-phase approach to handle in an original way the so-called *hidden phase problem*.

Index Terms— Segmentation, Optimization, Mumford-Shah functional, Multi-phase level sets, Hidden phase

1. INTRODUCTION

The active contour without edges (ACWE) formulation of Chan and Vese [1] is a popular variational segmentation approach that consists in minimizing an energy that is a weighted combination of a data fidelity term which measures the homogeneity of the image partition regions (phases) and a regularization term which forces the contours of the regions to remain smooth. The ACWE was introduced as a simplified version of the Mumford and Shah (MS) segmentation model [2], which remains unsolved in its general form (see [3, 4] for a detailed discussion on the MS conjecture). The simplification relies on the use of a piecewise constant model for the appearance of the objects in the image. Indeed for a piece-wise constant segmentation model, a minimizer of the simplified Mumford-Shah energy functional exists for any finite number of regions.

The ACWE exploits a continuous formulation of the optimization process, and uses level-set functions [5] to encode the dynamic evolution of the partition interfaces toward an optimal configuration. The use of level set functions enables topological changes of the partition which lead to a very flexible numerical implementation: arbitrary configuration of the initial partition (for example with multiple cylinders as in [1]), straightforward implementation in N-dimensions and straightforward extension of the formalism to more than two phases [6]. Regarding level-set based multiphase segmentation frameworks, related to the ACWE energy functional, three families of approaches have been studied: (1) use of multiple independent level set functions (i.e., N functions for N phases) as in [7], [8], (2) use of joint level set functions (i.e., N functions for 2^N phases), as in [6], [9], and (3) use of a single *multilayer* level set function (i.e., 1 level set function for N phases) as in [10]. The multilayer approach benefits from a smaller computational cost than when using multiple level set functions but forces a nested structure of the segmented regions. In this work we focus on the joint level set formulation (cf. (2)) defining 2^N phases as the intersection of the positive and negative parts of N level set functions leading to a complete non-overlapping partition of the image.

The possibility to use a generic segmentation framework with an arbitrary finite number of phases is very appealing in multiple applications, including brain MRI segmentation as in [11] where the ACWE performance was compared to other segmentation methods and [12] where different homogeneity measures were evaluated for the brain segmentation task. The multi-phase and multi-channel case was recently addressed in [13, 14].

On the other hand, the ACWE segmentation framework has not been pursued as actively as discrete counterparts such as the Markov Random Fields. We believe that the reason lies in numerical difficulties encountered in the standard implementation from [6] that can lead to too much sensitivity of the algorithm with respect to the initialization and the parameterization and even failure of the segmentation process, especially when manipulating more than two phases.

This paper proposes a generic re-writting of the Euler-Lagrange dynamic equations, identifies a numerical problem linked to the level-set implementation, which is called the *hidden phase problem*, and proposes a solution via a modification of the numerical implementation, involving a new permutation method.

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2. REWRITTING OF THE MULTI-PHASE ACWE

The ACWE segmentation exploits a region-based energy functional, which characterizes homogeneous regions through their average intensity values, combined with a regularization term to ensure smoothness of the contours. For a given image u_0 defined on Ω, the segmentation task consists in finding P regions R_j such that $\cup_{j=1}^{P} R_j = \Omega$, with associated contours ∂R_j and average pixel intensities c^j, minimizing the energy:

$$E = \sum_{j=1}^{P} [\lambda \int_{R_j} (u_0(x) - c^j)^2 dx + \nu \cdot \text{Length}(\partial R_j)], \quad (1)$$

which is the functional associated to the "minimal partition problem" proposed in [2], and is a restriction of the general MS functional to the piecewise constant case. The first term is the data fidelity term and will be called the homogeneity measure, while the second term is the regularization term. The parameters λ and ν are positive scalar weights.

The multi-phase ACWE implementation introduced in [6] encodes contours in level set functions and evolves N level set functions that define up to 2^N phases (or regions), by considering the possible intersections of all positive and negative regions of each level set function. To segment four phases, two level set functions $\Phi = (\phi_1, \phi_2)$ and the average values $c = c^{i,j}$ for $i, j = 0, 1$ are defined, according to the sign of ϕ_1 and ϕ_2, respectively. Then, the energy writes:

$$\begin{aligned} F(c, \Phi) = \lambda_{1,1} \iint_\Omega (u_0(x) - c^{1,1})^2 H(\phi_1(x)) H(\phi_2(x)) dx \\ + \lambda_{1,0} \iint_\Omega (u_0(x) - c^{1,0})^2 H(\phi_1(x))(1 - H(\phi_2(x))) dx \\ + \lambda_{0,1} \iint_\Omega (u_0(x) - c^{0,1})^2 (1 - H(\phi_1(x))) H(\phi_2(x)) dx \\ + \lambda_{0,0} \iint_\Omega (u_0(x) - c^{0,0})^2 (1 - H(\phi_1(x)))(1 - H(\phi_2(x))) dx \\ + \nu \iint_\Omega (\mid \nabla H(\phi_1(x)) \mid + \mid \nabla H(\phi_2(x)) \mid) \, dx, \end{aligned} \quad (2)$$

where H is the Heaviside function. This level-set implementation shows several good properties: (1) it is computationally more efficient than using 2^N level set functions, (2) extension from 2 to N phases is straightforward (which is useful for segmenting a large number of objects), (3) it guarantees to provide a partition of the image, with no vacuum nor overlap. The drawback of defining phases implicitly, from the intersections of level set functions, is twofold: (1) a change on any of these functions may affect all phases. Reciprocally, changing one phase in a certain way may require to evolve multiple level set functions; (2) the length of the region contours is hard to compute, as pointed out in [1].

Formulation of the energy functional and the Euler-Lagrange dynamic equations quickly become tedious to write in the original form proposed in [6] when more than two level set functions are used. In order to generalize these formulas to the 2^N-phase case ($N > 2$), we first introduce new notations which can be used to implement a code in which N can be chosen arbitrarily. These notations establish a correspondence between the signs of the level set functions, represented by 1 (positive) or 0 (negative) and a representation of the phases with binary numbers.

In the multi-phase model with N level set functions, each phase $j = 0, \ldots, 2^N - 1$ is defined as the intersection between N sets of pixels, each of which corresponding either to the positive or negative part of the level set functions $\{\phi_i\}_{i=1,2,..,N}$. Defining the phases therefore amounts to knowing the signs of each level set function anywhere in the image domain Ω. For that matter, as well as to shorten the notations, we introduce three definitions.

Given N level set functions $\{\phi_i\}_{i=1,2,..,N}$, with $\phi_i : \Omega \mapsto \mathbb{R}$ the first definition assigns an index j to a phase based on the sign of the N level set functions, encoded in a N-digits binary representation. For example for $N = 3$, the phase $j = 101$ in binary notations includes all the pixels where: $\phi_1 > 0, \phi_2 < 0$ and $\phi_3 > 0$, and is encoded with the scalar index $j = 5$.

Definition 1 *For a positive integer N, let the integer $j \in [0, 2^N - 1]$. For each $1 \leq m \leq N$, let $B^j(m)$ be the m^{th} bit (binary digit) of j, so that the vector B^j encodes the binary representation of j. Reciprocally, given a binary vector B of length N, we have $j = \sum_{k=0}^{N-1} 2^k B^j(k+1)$.*

We then define the characteristic function χ_j of the phase j, according to the sign of each level set function Φ_l encoded in the l^{th} bit of the vector B^j.

Definition 2 *Given a positive integer N, for all $0 \leq j \leq 2^N - 1$, we define the vectors B^j in $\{0, 1\}^N$ according to Definition 1. For all the level-set functions ϕ_l, $1 \leq l \leq N$, we define the scalar functions χ_l^j as:*

$$\chi_l^j = \begin{cases} H(\phi_l) & \text{if } B^j(l) = 1 \\ 1 - H(\phi_l) & \text{if } B^j(l) = 0. \end{cases} \quad (3)$$

We define the characteristic function χ_j of the phase j as:

$$\chi_j = \prod_{1 \leq l \leq N} \chi_l^j. \quad (4)$$

Finally, we define the characteristic function $\hat{\chi}_{l,h^1,h^2}$ to be used in the dynamical equation that evolves the level set function ϕ_l, based on the difference of the homogeneity measures inside and outside the set $\{x \in \Omega \mid \phi_l(x) = 0\}$, for $h_1 = 1, \ldots, l-1$ and $h_2 = l+1, \ldots, N$.

Definition 3 *Given a positive integer N, for all $1 \leq p \leq N$, let $h_l \in [0, 2^{p-1} - 1]$ and $h_r \in [0, 2^{N-p} - 1]$ and let the vectors B^{h_l} and B^{h_r} of length $p-1$ and $N-p$ be defined from Definition 1. For all the level-set functions ϕ_i, $1 \leq i \leq N$ and $i \neq p$, we define the function $\hat{\chi}_i^{h_*}$ (with $h_* = h_l$ or h_r) via:*

$$\hat{\chi}_i^{h_*} = \begin{cases} H(\phi_i) & \text{if } B^{h_*}(i) = 1 \\ 1 - H(\phi_i) & \text{if } B^{h_*}(i) = 0. \end{cases} \quad (5)$$

Then we define the characteristic function of the phase defined, without taking into account ϕ_p, through the binary representation $[B^{h_r}, B^{h_l}]$, where $[.,.]$ is the concatenation operator, as the product :

$$\hat{\chi}_{p,h_l,h_r} = \prod_{1\leq i\leq N, i<p} \hat{\chi}_i^{h_l} \prod_{1\leq i\leq N, i>p} \hat{\chi}_i^{h_r}. \quad (6)$$

For a given level set function ϕ_p, each $\hat{\chi}_{p,h_l,h_r}$ (with respect to all possible h_l and h_r values) will be used to define the positive phase defined with the binary number $B_+^{l,h_l,h_r} = [B^{h_l}, 1, B^{h_2}]$ and the negative phase defined with the binary number $B_-^{l,h^l,h^r} = [B^{h_l}, 0, B^{h_r}]$. These definitions therefore provide a generic formalism to implement the multiphase ACWE functional minimization with an arbitrary finite number of level-set functions. The energy (1) to minimize is rewritten as follows. Given N level set functions ϕ_l, $1 \leq l \leq N$, let $\Phi = (\phi_1, \ldots, \phi_N)$ and suppose that the homogeneity measure associated with phase j ($0 \leq j \leq 2^N - 1$) is defined by:

$$z^j(x) = \lambda_j (u_0(x) - c^j)^2, \quad (7)$$

where c^j is the average pixel intensity within phase j. A recommended setting for the weight parameter λ_j is $\lambda_j = \frac{1}{K(u_0)^2}$ where $K(u_0) = \max_{y\in\Omega} u_0(y) - \min_{y\in\Omega} u_0(y)$ is the contrast of u_0 (independent of j), so that $z^j \in [0,1]$. Let now $Z = (z^1, \ldots, z^{(2^N-1)})$, then the energy writes:

$$F(Z,\Phi) = \nu \sum_{l=1}^{N} \int_\Omega \mid \nabla H(\phi_l(x)) \mid dx + \sum_{j=0}^{2^N-1} \int_\Omega z^j(x)\chi_j(x)dx.$$

To minimize this expression, we embed the associated Euler-Lagrange equations with respect to the level set functions $\phi_i(x)$ in a dynamical scheme with artificial time t to evolve $\phi_i(t,x)$. Given the initial functions $\phi_i(0,x)$ for $1 \leq i \leq N$ and $x \in \Omega$, we compute the values of the vector Z as the average intensity values inside each phase (characterized through the χ_i) and update ϕ_i as follows:

$$\frac{\partial \phi_i}{\partial t} = \delta_\epsilon \Big\{ \nu \text{div}\left(\frac{\nabla \phi_i}{\mid \nabla \phi_i \mid}\right) - \sum_{\substack{0\leq h_l \leq 2^{i-1}-1 \\ 0\leq h_r \leq 2^{N-i}-1}} \left[z^{\hat{B}_+^{i,h_l,h_r}} - z^{\hat{B}_-^{i,h_l,h_r}} \right] \hat{\chi}_{i,h_l,h_r} \Big\}, \quad (8)$$

where δ_ϵ corresponds to an approximation of the Dirac mass as in [1].

3. HIDDEN-PHASE PROBLEM AND PROPOSED SOLUTION

Problem statement. Now that we have rewritten the Euler-Lagrange equations of individual level set functions ϕ_i in a generic form, a close look at these evolution equations indicates a problem that needs to be addressed: for each level set function, the decision to increase or decrease its value at a given pixel is taken under the assumption that the sign of the other functions remains unchanged. Hence, for a pixel that currently belongs to a given phase, only N transitions of phases are explored (one per switch of the sign of the individual level set functions at the pixel location) out of the $2^N - 1$ possible phases. As a consequence, a pixel which is not currently classified in the optimal phase cannot always be moved to the best phase (i.e., the phase whose associated mean value is closer to the intensity value of the pixel). We call this problem the *hidden-phase* problem.

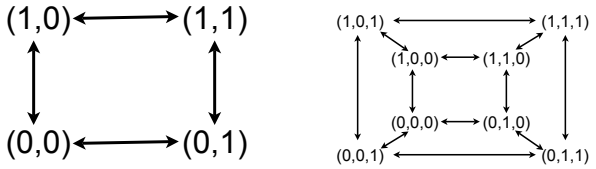

Fig. 1. Diagram of phase transition in the 4-phase model (left) and 8-phase model (right).

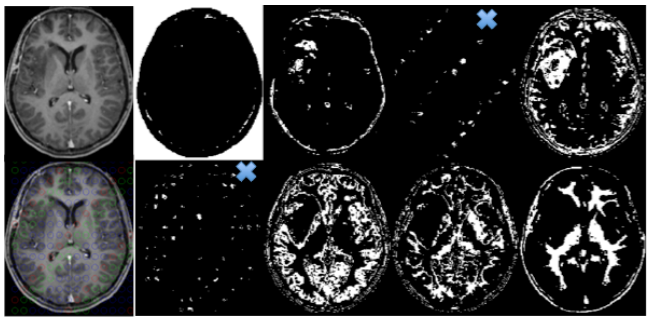

Fig. 2. Segmentation of a brain MRI (top-left) into 8 phases as initialized with 3 sets of cylinders (bottom-left), and an illustration of the hidden phase problem (8 figures on the rigth) preventing correct exploitation of 2 phases, marked with a cross.

The diagram of Fig. 1 summarizes this limitation with respect to the possible transitions that assign a given pixel from one phase to the others. An arrow between two phases indicates that a pixel can be swapped between the two phases, as long as this decreases the global energy. In the 4-phase case, each phase only "sees" two out of the three other phases and 2-step transitions might be needed to correctly segment a given pixel. The phase transition diagram for the 8-phase model is provided in the same figure. The 8-phase model suffers even more severely from this problem as the evolution of a given level set function can only drive a pixel to three other phases, leaving out four phases not directly accessible. In this case, the correct segmentation of a pixel may require up to three transitions, which must all successively decrease the energy. The existence of such a path between the phases becomes very unlikely and the final segmentation highly depends on the initialization.

The hidden phase problem can lead to the presence of patterns in the final segmentation, created by pixels which could not be moved to the appropriate phase, as illustrated in Fig. 2. These patterns are typically reminiscent of the initial shapes. Increasing the regularization parameter to get rid of those patterns is not acceptable as it would likely erase small objects from the segmentation. We now introduce an original algorithmic solution to authorize more transitions between phases and avoid the hidden phase problem.

Algorithmic solution to the hidden phase problem. Based on the Euler-Lagrange optimization, a phase transition of a pixel over a single iteration can only decrease the energy. We propose to preserve this behavior while ensuring that a given pixel "sees" all the phases over the course of the iterations, via permutation of some selected phases, in specific sequences described below. Permutation is performed via inversion of the sign of some given level set function(s) ϕ_i in a region where specific constraints on the sign of ϕ_j apply ($i \neq j$). For example a permutation of the phases $(1,1)$ and $(0,1)$ is performed via inverting the sign of ϕ_1 where $\phi_2 \geq 0$. In order to preserve the smoothness of the contours, the permutation requires a reinitialization of the level set functions involved. Permutations only need to be applied after a fixed number of iterations S, so that contours have evolved enough to let patterns due to the hidden phase problem appear. This generally took less than ten iterations in our experiments.

For the 4-phase implementation, we permute any couple of phases which are in direct relation with each other (see Fig. 1), e.g. $(1,1)$ and $(0,1)$. Hence, transitions from $(1,0)$ to $(0,1)$ and from $(1,1)$ to $(0,0)$ now become possible.

The proposed permutation method can be implemented for 8-phases as well, though it becomes a little more complex. To enable each phase to "see" any other one in the course of the iterations we optimize the inner and outer squares from Fig. 1 separately, as in the 4-phase case, by permuting phases $(1,1,0)$ and $(0,1,0)$ for the inner square and $(1,1,1)$ and $(0,1,1)$ for the outer square. At this point, after S iterations prior to the permutation and S iterations after, all the transitions between phases $(\cdot,\cdot,0)$ and between phases $(\cdot,\cdot,1)$ have been allowed and only the links between the first and second set of phases (inner and outer squares) remain to be established. For an inner-square phase, there is only one outer-square phase that is currently accessible. Therefore, applying a circular permutation on the inner-square phases will permit all inner-square phases to be directly linked to a different outer-square phase. If we apply the same circular permutation three times, all outer-square phases will have been "seen" by any of the inner-square phases. Therefore, we need $3S$ iterations to perform those three circular permutations leading to a total of $5S$ iterations. The proposed permutation method could be generalized for $N > 3$ but we do not detail it, since eight phases are generally sufficient for most multi-object segmentation tasks, as in brain imaging.

The advantage of the proposed permutation method is that we remain in the exact same framework as the original multi-phase ACWE, except that the solution obtained before permutation can be considered as an initialization for the next S iterations, which are computed after permutation and reinitialization. This implies that at each iteration, we are guaranteed to decrease the energy, and that a pixel can never be classified, even temporarily, in a phase whose associated mean intensity value is worse than before. A reinitialization is only performed every S iterations, which does not add significant computation burdens, while ensuring numerical stability of the evolution of the level-set functions evolution away from the zero-levels.

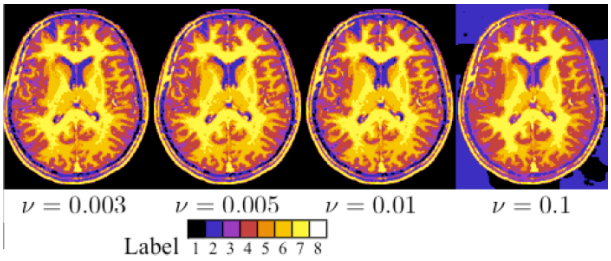

Fig. 3. 2D segmentations on a T1-weighted brain MRI (SPGR) with the 8-phase ACWE free of the hidden phase problem, for different values of the regularization parameter ν. Colormap corresponds to the mean intensity value rank of each phase.

4. EXPERIMENTS & DISCUSSION

We illustrate in Fig. 3 the robust behavior of the 8-phase ACWE segmentation on the T1-weighted brain MRI image of a patient with a brain tumor, from the Centre Hospitalier Sainte Anne (Paris, France), illustrated in Fig. 2, for different values of the regularization parameter ν. The initialization consisted of N sets of cylinders which defined the zero-level of each of the N level set functions as in [6]. The hidden phase problem was clearly avoided and the influence of the regularization parameter can be clearly appreciated visually (similar results have been obtained on other images). The 8-phase ACWE used two cycles of permutations, and $2 \times 5S$ iterations overall. This total number of iterations with the proposed permutation method remains smaller than the number of iterations reported with the original implementation of Chan and Vese that suffers from the hidden phase problem. Nevertheless, the segmentation results in Fig. 3 reveal that two phases can be used to represent a homogeneous tissue or the background and that a single phase can include two different structures (eg. white matter and enhanced vessels). These limitations, which are not related to the hidden phase problem, lead to the open question of the automated identification and then enforcement of the number of phases to use for a given segmentation task, which was recently discussed in [16] or in [17].

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