On Some Associations Between Mathematical Morphology and Artificial Intelligence

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Abstract. This paper aims at providing an overview of the use of mathematical morphology, in its algebraic setting, in several fields of artificial intelligence (AI). Three domains of AI will be covered. In the first domain, mathematical morphology operators will be expressed in some logics (propositional, modal, description logics) to answer typical questions in knowledge representation and reasoning, such as revision, fusion, explanatory relations, satisfying usual postulates. In the second domain, spatial reasoning will benefit from spatial relations modeled using fuzzy sets and morphological operators, with applications in model-based image understanding. In the third domain, interactions between mathematical morphology and deep learning will be detailed. Morphological neural networks were introduced as an alternative to classical architectures, yielding a new geometry in decision surfaces. Deep networks were also trained to learn morphological operators and pipelines, and morphological algorithms were used as companion tools to machine learning, for pre/post processing or even regularization purposes. These ideas have known a large resurgence in the last few years and new ones are emerging.

Keywords: Mathematical morphology · Artificial intelligence · Lattice · Logics · Spatial reasoning · Fuzzy sets · Neural networks · Deep learning

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1 Introduction

This paper aims at highlighting the usefulness of mathematical morphology in artificial intelligence (AI). To this end, we restrict ourselves to the deterministic setting and to increasing operators. The underlying structure of complete lattices allows applying basic operators to various settings, and this is the bridge we establish between the so far rather disconnected domains of mathematical morphology and AI formalisms. Concrete operators depending on structuring elements will provide simple and intuitive examples. On the AI side, we address several of its components, from purely symbolic approaches to machine learning.

In Sect. 2, we illustrate links between mathematical morphology and logics. In propositional logics, considering the lattice of formulas, morphological operators will act on formulas (and on their models). Such examples will be described in several logics. We will then show how they can be used in typical reasoning problems in AI, to define revision operators, merging operators, or explanatory relations. In Sect. 3, mathematical morphology will be shown to be helpful for spatial reasoning. Spatial reasoning aims at modeling spatial entities and spatial relations to reason about them. A typical example is the problem of model-based image understanding. Models of a scene usually involve spatial relations to provide information on the structure of the scene and on the spatial arrangement of the objects it contains. Moreover, such relations allow disambiguating objects with similar shapes and appearances, and are more robust to deformations or pathological cases. Mathematical morphology is then useful, combined with fuzzy sets, to model such spatial relations, taking into account their intrinsic vagueness (e.g. left to, close to), and to compute them efficiently. These models can then be used in spatial reasoning processes. Finally in Sect. 4, we will move to machine learning methods in AI, and combine mathematical morphology and deep learning. Interactions between mathematical morphology and deep learning have been investigated since the 1980s across several aspects. Besides, they have known a large resurgence in the last few years, and new ones are emerging. We shall give an overview of this trend.

This paper is based on a tutorial taught by some of the authors at ECAI 2020. It is an overview, relying on the existing literature and previous works by the authors, and paving the way for future research at the cross-road of mathematical morphology and artificial intelligence.

2 Mathematical Morphology and Logics

In this section, we first show how basic morphological operators can be applied to logical formulas, and then use them to address typical reasoning problems in artificial intelligence.

Propositional Logic. Let us start with propositional logic, as originally proposed in [13]. We assume that the language is defined by a set V of variables (here assumed to be finite), denoted by $a, b...$, and the standard connectives

$\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\rightarrow$ (implication). Formulas are built on this language. The consequence relation is denoted by $\vdash$. Tautology and antilogy are denoted by $\top$ and $\bot$, respectively. A world or interpretation is an assignment of truth values to all variables. Ω denotes the set of all worlds. The set of models of a formula φ is denoted by $[\![\varphi]\!]$, and is a subset of Ω, defined as the set of worlds in which the formula is true. Since a formula is semantically characterized by its set of models, it is equivalent to consider the lattice of the set of formulas (up to syntactic equivalence) and the lattice $(\mathcal{P}(\Omega), \subseteq)$ where $\mathcal{P}(\Omega)$ is the set of subsets of Ω. Algebraic morphological operators are then defined as in any lattice. Now, the notion of structuring element can be formalized in this setting as a binary relation between worlds, and we will note for a relation B: $\omega' \in B_\omega$ iff (ω, ω') belongs to the relation. The inverse relation is denoted by $\check{B}$ ($\omega \in \check{B}_{\omega'}$ iff $\omega' \in B_\omega$). Then, the morphological dilation of a formula φ with a structuring element B is simply defined via the semantic as:

$$[\![\delta_B(\varphi)]\!] = \delta_B([\![\varphi]\!]) = \{\omega \in \Omega \mid \check{B}_\omega \cap [\![\varphi]\!] \neq \emptyset\},$$

In a similar way, the morphological erosion is defined as:

$$[\![\varepsilon_B(\varphi)]\!] = \varepsilon_B([\![\varphi]\!]) = \{\omega \in \Omega \mid B_\omega \subseteq [\![\varphi]\!]\}.$$

Let us illustrate these ideas on a simple example, where the relation B between worlds is defined as a neighborhood relation, *e.g.* based on a threshold on a Hamming distance d_H between worlds. Let $\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c)$, and B be the ball of radius 1 of the Hamming distance: $B_\omega = \{\omega' \in \Omega \mid d_H(\omega, \omega') \leq 1\}$, *i.e.* B_ω comprises all worlds where at most one variable is instantiated differently from ω. The dilation of φ by B is then $\delta(\varphi) = (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee c)$.

Many other operators from mathematical morphology can be defined in a similar way, and can be used to filter sets of models, making the operations described next more robust to noise or outliers for instance, to segment the main parts of a formula, etc.

Reasoning in AI Using Morphological Operators. With the help of these operators of dilation and erosion, we can successfully tackle some important reasoning aspects in AI, such as revision, merging (or fusion), abduction (explanatory reasoning), mediation..., which can find very concrete solutions in this morphological framework [14,15,17,18]. Another interesting feature is that these solutions, while simple and tractable, satisfy the properties usually required in such reasoning problems. Let us mention three examples.

Belief Revision. Let us assume that we have a set of preferences or beliefs, represented by one formula φ, as in [32]. If a new information is available, modeled as a formula ψ, the initial set should be revised to account for this new information [4]. This revision operation, denoted by $\varphi \circ \psi$ is usually assumed to induce a minimal change on the initial set of beliefs. A very simple concrete form for $\circ$ is to dilate the initial preferences or beliefs, until they become consistent with

ψ, i.e. $\varphi \circ \psi = \delta^n(\varphi) \wedge \psi$, with $n = \min\{k \in \mathbb{N} \mid \delta^k(\varphi) \wedge \psi \text{ is consistent}\}$. Taking the minimum size of dilation achieving consistency directly models the idea of minimal change. Interestingly enough, it can be proved that this particular revision operator satisfies the AGM postulates [4] in Katsuno and Mendelzon's model [32], which are widely considered as the postulates every revision operator should satisfy. Hence, the dilation based approach allows for a concrete computation of the revision, satisfying both the AGM postulates and the minimal change constraint.

Belief Merging. Let us now consider that m sets of beliefs or preferences are represented by logical formulas $\{\varphi_1 ... \varphi_m\}$ (which is a multi-set). Again, dilation can be the basis of the definition of several operators for logical fusion of these belief sets, under integrity constraints encoded as a propositional formula μ. Typically, each set of beliefs can be dilated until their conjunction with the constraint is consistent, e.g. $\Delta_\mu^{\max}(\varphi_1, ..., \varphi_m) = \delta^n(\varphi_1) \wedge \delta^n(\varphi_2) \wedge ... \wedge \delta^n(\varphi_m) \wedge \mu$, where $n = \min\{k \in \mathbb{N} \mid \delta^k(\varphi_1) \wedge ... \wedge \delta^k(\varphi_m) \wedge \mu \text{ is consistent}\}$, or $\Delta_\mu^{\Sigma}(\varphi_1, ..., \varphi_m) = \bigvee_{(n_1,...,n_m)} \delta^{n_1}(\varphi_1) \wedge \delta^{n_2}(\varphi_2) \wedge ... \wedge \delta^{n_m}(\varphi_m) \wedge \mu$ where $\sum_{i=1}^m n_i$ is minimal with $\delta^{n_1}(\varphi_1) \wedge \delta^{n_2}(\varphi_2) \wedge ... \wedge \delta^{n_m}(\varphi_m) \wedge \mu$ consistent. These two definitions correspond to merging operators introduced in [34], and satisfy the rationality postulates such operators should satisfy. An example is illustrated, by representing models as sets, in Fig. 1 (left).

Abduction. Abductive reasoning belongs to the now popular domain of explainable AI, and aims at finding the "best" explanation γ to an observation α, according to a knowledge base Σ. In a logical setting, Σ is a set of formulas, and γ and α are formulas. The problem is then expressed as: $\Sigma \cup \{\gamma\} \vdash \alpha$. Again, we expect γ to satisfy a number or properties, expressed as rationality postulates in [42,43]. The general idea of using mathematical morphology in this context is to find the most central models in Σ or in $\Sigma \wedge \alpha$, satisfying α. Note that reducing the set of models amounts to adding formulas to Σ. This idea can be implemented in a simple way by using successive erosions, *e.g.* as:

$$\alpha \rhd^{\ell} \gamma \overset{def}{\Leftrightarrow} \gamma \vdash_{\Sigma} \varepsilon_{\ell}(\Sigma \wedge \alpha)\ ; \text{ or } \alpha \rhd^{\ell c} \gamma \overset{def}{\Leftrightarrow} \gamma \vdash_{\Sigma} \varepsilon_{\ell c}(\Sigma, \alpha) \wedge \alpha$$

where ε_ℓ denotes the last erosion before obtaining an empty set, and $\varepsilon_{\ell c}$ denotes the last erosion which is still consistent with α. An example is illustrated in Fig. 1 (right). A useful feature of this approach is that different types of explanations can be obtained by appropriate choices of the structuring element used in the erosions. For instance if $\Sigma = \{a \rightarrow c, b \rightarrow c\}$ and $\alpha = c$, then depending on the meaning of a, b, c, we can seek a disjunctive explanation of c $(a \vee b)$, or a conjunctive one $(a \wedge b)$, or an exclusive disjunction $((a \wedge \neg b) \vee (\neg a \wedge b))$ [14].

Other Logics. The ideas described above have been extended to several other logics, more expressive than propositional logic. We just mention some of these extensions here:

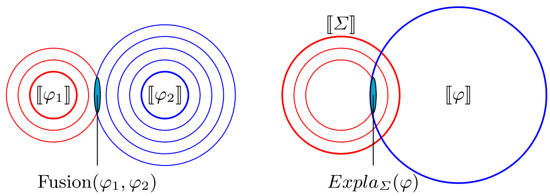

Fig. 1. Fusion based on dilations (left) and explanation based on erosions (right).

- Dilation of formulas expressed in first order logic has been used in [28,29] for application to merging, along with an efficient implementation using decision diagrams.
- In modal logic, the two modalities modeling possibility $\Diamond$ and necessity $\Box$ can be expressed as dilation and erosion, respectively [16]. From a semantic point of view, the accessibility relation, expressing a binary relation between worlds, can be considered as a structuring element. It is then natural to consider that $\Box\varphi$ is satisfied in a world ω if φ is satisfied in all worlds accessible from ω, corresponding to the erosion by this structuring element, while $\Diamond\varphi$ is satisfied in ω if φ is satisfied in at least one world accessible from ω, hence corresponding to the dilation.
- In description logic δ and ε can be naturally included in the logic as binary predicates [6,31], and are thus involved in ontological reasoning.
- A generalization of all the above was developed in the abstract framework of satisfaction systems and institutions, which encompasses many logics [1–3]. This allows extending the revision operator described above in propositional logic to a revision operator acting in any logic, based on a notion of relaxation (slightly different from a dilation). Similarly, abductive reasoning can be performed based on notions of cutting and retraction, similar to erosions.

3 Mathematical Morphology for Spatial Reasoning

In this section, we illustrate how mathematical morphology can be used for spatial reasoning in various settings. Spatial reasoning is defined as the domain of knowledge representation on spatial entities and spatial relationships, and reasoning on them. Spatial entities can be represented as abstract formulas in a logical (symbolic, qualitative) setting, as regions or keypoints in a quantitative setting, or as fuzzy regions in the semi-qualitative setting of fuzzy sets (*i.e.* still a deterministic setting). On the symbolic side, spatial relations can be represented as formulas, connectives, modalities or predicates in a logical setting. On the numerical side, they are best represented using fuzzy models, in order to account for their intrinsic imprecision (*e.g.* "to the right of", "close to").

Spatial Reasoning in a Qualitative Setting Using Morpho-Logic. Let us first consider the qualitative setting, using various logics. As a first example, let us consider abductive reasoning, as introduced in Sect. 2. Image understanding can be expressed as an explanatory process, which aims at providing the "best" explanation γ to the observations $\mathcal{O}$ according to a knowledge base $\mathcal{K}$ [6], *i.e.* such that $\mathcal{K} \cup \{\gamma\} \vdash \mathcal{O}$. Observations can be images, or results of some image analysis process (e.g. segmentation of some structures in the images). The knowledge base $\mathcal{K}$ models expert knowledge on the domain, on the structures present in the scene and on their relations (contrast, spatial relations...). It can be expressed in description logic for instance. A solution to the abduction problem consists in translating knowledge and observation in a lattice of concepts, and by applying erosions in this lattice to find the best explanation according to a minimality criterion [6]. Another algorithmic solution relies on tableau methods [54], an algorithmic way to satisfiability problems, where formulas are developed in different branches until inconsistencies are found. In the example in Fig. 2 (left), a MRI brain image with a tumor can be interpreted, at a higher level, using an anatomical knowledge base, as "Peripheral Small Deforming Tumoral Brain".

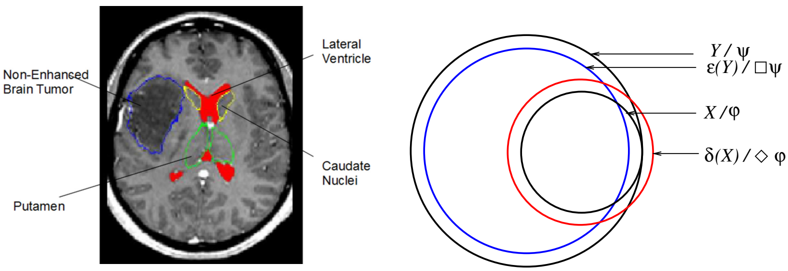

Fig. 2. Left: pathological brain with a tumor. Finding a high level interpretation of the image can be formalized as an abduction problem, where the knowledge base contains expert knowledge and the observation is the image and segmentation results. Right: Tangential part from morphological operators (X and Y are models of formulas φ and ψ, respectively).

Let us now consider modal morpho-logic, where the two modalities are defined as erosion and dilation (*i.e.* $\Box \equiv \varepsilon$ and $\Diamond \equiv \delta$), and consider the domain of mereotopology, specifically the Region Connection Calculus (RCC) formalism [45]. In this theory, several topological relations are defined from a connection predicate, in first order logic. Modal morpho-logic leads to simpler and decidable expressions of some of these relations. Let us provide a few examples, where φ and ψ are formulas representing abstract spatial entities:

- φ is a tangential part of ψ iff $\varphi \rightarrow \psi$ and $\Diamond\varphi \wedge \neg\psi \not\rightarrow \bot$ (or $\varphi \rightarrow \psi$ and $\varphi \wedge \neg\Box\psi \not\rightarrow \bot$). A simple model in the 2D space of such a relation is illustrated in Fig. 2 (right).
- φ is a non tangential part of ψ iff $\Diamond\varphi \rightarrow \psi$ (or $\varphi \rightarrow \Box\psi$).
- φ and ψ are externally connected (adjacent) iff $\varphi \wedge \psi \rightarrow \bot$ and $\Diamond\varphi \wedge \psi \not\rightarrow \bot$ (or $\varphi \wedge \Diamond\psi \not\rightarrow \bot$).

Further links between mathematical morphology and RCC can be found in [10,12,35].

Semi-qualitative Framework Using Fuzzy Modeling of Spatial Relations. When imprecision on knowledge and on data has to be taken into account, a semi-qualitative framework is best appropriate, such as fuzzy sets theory. In this theory, every piece of information becomes a matter of degree, and the membership of an element to a set, the degree to which some elements satisfy a relation, the truth value of a logical formula, etc. are values in $[0, 1]$. Note that what is most important is the ranking between different values, rather than their absolute value. This theory offers many tools for information representation and processing [26]. One of the problems to be addressed when reasoning on both qualitative or symbolic knowledge and on numerical data is the so-called semantic gap, between abstract concepts and concrete information extracted from data (e.g. images). The notion of linguistic variable [55] is then useful to establish links between a concept and its representation in a specific concrete domain. Here again, mathematical morphology, extended to handle fuzzy sets [8], can play an important role that we illustrate here on the modeling of spatial relations [7].

The main idea is to model the semantics of a spatial relation R expressed in a linguistic way as a fuzzy structuring element ν_R (i.e. a function from the spatial domain into $[0, 1]$). Then, dilating a reference object, possibly fuzzy, defined by its membership function μ, provides a fuzzy region of space where the value $\delta_{\nu_R}(\mu)(x)$ at each point x represents the degree to which relation R to μ is satisfied at this point. The degree to which another object satisfies relation R to μ can then be computed from the aggregation, using some fuzzy fusion operator, of the values $\delta_{\nu_R}(\mu)(y)$ for all points y of this second object. Several relations can be modeled according to this principle, such as topological relations, directions, distances, as well as more complex relations (between, along, parallel, aligned...). A useful feature of fuzzy representations is that a relation and its degree of satisfaction can be represented as a number, a fuzzy number, an interval, a distribution, a spatial fuzzy set, etc., in a same unifying framework.

This framework can be the basis of spatial reasoning, for example for structural model based image understanding. Indeed, modeling explicitly the structural knowledge we may have on a scene helps recognizing individual structures (disambiguating them in case of similar shape and appearance), as well as their global organization. This can be achieved using various methods, each having two main components: knowledge representation and reasoning. Fuzzy models of objects and relations can enhance qualitative representations (logical formulas, knowledge bases, ontologies), and then be used in logical reasoning, including morpho-logic (for instance, the set of models of a formula becomes a fuzzy set). They can serve as attributes in structural representations such as graphs, hypergraphs, conceptual graphs. Reasoning then relies on matching, sequential graph traversal to guide the exploration of an image, constraint satisfaction problems, etc. (see *e.g.* [9,11] and the references therein). From the interpretation results, it is then possible to go back to the initial language of the domain to provide linguistic descriptions of the image content, as in the previous example on brain image interpretation.

4 Mathematical Morphology and Neural Networks

We now move to a different branch of AI, namely machine learning based on neural networks, which has tremendously grown in the past years. As a matter of fact, there has been an increasing effort devoted to the combination of mathematical morphology and neural network frameworks. More specifically, morphological operations can be integrated as efficient pre- or post-processing tools within machine/deep learning processing frameworks. Further, the structural similarity between neuron operations (weighted linear combination of input values, potentially mapped by non-linear activation functions) and elementary morphological operations such as erosion and dilation makes it tempting to substitute the former by the latter, resulting in morphological perceptrons and morphological layers. This section reviews these major lines of research.

Mathematical Morphology as a Pre/Post-Processing Step. A well-known drawback of deep convolutional neural networks (CNNs) is their poor ability to segment very thin structures, and their sensibility to noise and contrast [20,25]. Integrating mathematical morphology as pre-processing has improved the results of several classical CNN architectures on such problems. For example, the use of the morphological top-hat helps to enhance the very small structures of medical images. In [53], the top-hat is used in one of the three channels of a VGG input to segment small white matter hyperintensities, while in [22], the top-hat results are directly fed to the networks and proved their usefulness by guiding the networks (here ConvNet and Mask R-CNN) to focus on chosen parts of the image (here knee meniscus tear). In [24], a geodesic reconstruction is performed to inject topological (global) information into Whole Slide Imaging images (that are huge and heavy) before doing patches. This step improved the skin segmentation results of U-Net.

Mathematical morphology can also be helpful in post-processing steps. For example, to avoid heavy computation time and memory use, some 3D segmentation tasks can be treated as successive 2D segmentations. Without any spatial context, successive segmented slices can be disconnected or show aberrations. Mathematical morphology can hence be used as a regularization step to remove these abnormalities [44].

Morphological Neural Networks for Images. Morphological neural networks were introduced in the late 1980s with a definition of neurons as weighted rank filters [52] or, in a less general form, as performing dilations or erosions [23]. Replacing the linear perceptron's dot product by the non-linear max-plus and min-plus operators has induced a new geometry of decision surfaces, which we may refer to as *bounding box* geometry [46,49,50,56], and alternative (or complementary) strategies to gradient descent in networks training. Hybrid approaches mixing linear and morphological layers have also been developed for an even richer geometry [21,30,41,51,57], and dilation layers showed interesting pruning properties when located after linear layers [21,57]. The latter studies, however, only consider dense layers and are therefore little suited to image analysis, in contrast to convolutional networks which can handle large images.

Yet, translation invariant morphological architectures are a crucial issue. Indeed, the success of a morphological framework involving elementary operations (dilation and erosion) and their combinations (closing, opening, top-hat, etc.) often comes with a tedious trial-and-error setting to derive the optimal sequence of operations and their respective structuring elements. Deep CNNs are a potential solution to automatically learn this optimal sequence and the structuring elements. As a matter of fact, the weights of each layer filter could be interpreted as (non necessary flat) structuring elements, provided that the conventional convolution operation has been replaced by erosion or dilation, and that the layer has a way to learn which operation to use. Strictly speaking, one should then talk in that case about morphological (and no longer convolutional) neural network architectures.

In the early 2010s, when deep learning optimization tools were not as well diffused as today, attempts were made to overcome the non-differentiability of the min and max operations of erosions and dilations in convolutional-like approaches relying on stochastic gradient descent [37]. Based on the Counter-Harmonic Mean (CHM) [5,19], the so-called $PConv$ layer is not only smooth, but it can approximate non-flat dilations, erosions and classical convolutions, depending on their parameter p, which can be trained along with the kernel parameters. Recently, this idea was successfully applied to digit recognition tasks [38]. Even lately, another smooth approximation of min and max, *i.e.* the so-called LogSumExp function (also known as multivariate softplus), has been investigated in [48] to learn binary structuring elements, and extended to grayscale structuring elements in [47]. Finally, a last smooth version of min and max operations based on the α-softmax [36] function has been proposed and shown to outperform the classical $PConv$ layer in learning non flat structuring elements [33].

On the other hand, deep neural networks including non-smooth operators (the ReLU activation function, max and min pooling layers, to name a few) have been efficiently trained with stochastic gradient descent for years. Indeed, these operators are actually differentiable almost everywhere, and a descent direction can be defined even in their zero-measure non-smooth regions. Therefore, it is natural that translation-invariant morphological layers were recently optimized just as usual convolutional ones [27,39,40], that is, with stochastic gradient descent and back-propagation. In the latter studies, deep architectures including morphological layers were applied to classification, image denoising and restoration as well as edge detection. Two remarkable results are that the morphological trainable max-pooling improves significantly the classical max-pooling, and that independent morphological layers converge without constraint towards an adjunction.

From this brief review, some challenges clearly appear. First, the morphological networks we just mentioned are all way more shallow than the most popular classical architectures, and none of them compete with state-of-the-art CNNs on tasks like segmentation or classification on large-scale image datasets. Besides, successful architectures including morphological layers almost always contain classical convolutional layers. These two observations tend to indicate that insights are still needed regarding the representation power of purely morphological networks, as well as their optimization when many layers are stacked.

5 Conclusion

In this paper, we highlighted some links between mathematical morphology and different components of artificial intelligence, including symbolic AI for knowledge representation and reasoning in various logics, fuzzy sets for reasoning under uncertainty, and machine learning based on neural networks, which are all very active topics in AI. Future work on mathematical morphology and symbolic AI is planned in two ways: extend the "toolbox" of morpho-logic with other morphological operations to enrich both knowledge representation and reasoning, and enhance mathematical morphology with the inference power of logics. Both directions can be endowed with an uncertainty modeling layer, based on fuzzy sets theory. Similarly, a current trend in deep learning is to introduce knowledge in neural networks, which could be, in the future, modeled as morpho-logic or fuzzy spatial relations.

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