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On fuzzy distances and their use in image processing under imprecision

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Abstract

This paper proposes a classification of fuzzy distances with respect to the requirements needed for applications in image processing under imprecision. We distinguish, on the one hand, distances that basically compare only the membership functions representing the concerned fuzzy objects, and, on the other hand, distances that combine spatial distance between objects and membership functions. To our point of view, the second class of methods finds more general applications in image processing since these methods take into account both spatial information and information related to the imprecision attached to the image objects. New distances based on mathematical morphology are proposed in this second class.

Keywords: Fuzzy sets; Image processing; Fuzzy distances; Fuzzy mathematical morphology

1. Introduction

Fuzzy set theory finds in image processing a growing application domain. This may be explained not only by its ability to model the inherent imprecision of images together with expert knowledge, but also by the large and powerful toolbox it offers for dealing with spatial information under imprecision [1]. This is, in particular, highlighted when spatial structures or objects in the images are directly represented by fuzzy sets. A large set of image processing transformations involves the analysis of structures taking into account geometrical, topological, morphological, distance, connectivity and neighborhood information. In particular, pattern recognition is performed by considering not only the object to be recognized but also the context information provided by the other objects in the scene. The interest of fuzzy spatial relationships for representing and processing imprecise image content has been highlighted by several authors several years ago (see e.g. [2–4]).

Several set and geometrical measures and transformations have been generalized to fuzzy sets (e.g. [5,6]). Fuzzy topological and connectivity aspects have been introduced in [7]. Fuzzy morphological operators have been defined [8,9], which analyze relationships between fuzzy sets and fuzzy structuring elements by controlling the spatial extension of the transformations. Other fuzzy relationships like adjacency [10] or relative position [11–13] have also been developed recently.

Distance measures have been proposed in several works, and constitute the purpose of this paper. The most common approach consists in defining distances between two fuzzy sets. We will shortly review the proposed definitions, classify them depending on the type of information they convey, and propose some generalizations. The presentation given below is directly inspired by the classification proposed in [14], but adapted to image processing purposes, by underlining for each definition the type of image information on which it relies. One class of approaches is concerned only by the comparison of the membership functions and is widely

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addressed in the literature, while the second one introduces also spatial domain distances and constitutes the most original part of this paper. In Section 2, we first recall basic definitions and properties of proximity relationships and distances, and discuss the needs for image processing. Then we present the main methods that can be used for defining fuzzy distances. In Section 3 we propose a classification of distances relying only on the comparison of membership functions. In Section 4 we define original distances taking also spatial information into account. A simple illustrative example is given in Section 5. Finally, we provide a short discussion on the possible use of these two classes of distances in image processing and pattern recognition in Section 6.

2. Fuzzy distances and image processing

In this section we specify the context of this study and recall basic definitions that will be used in the following. We also present the principles underlying the main approaches for defining fuzzy distances. These principles will be instantiated under different forms in Sections 3 and 4.

2.1. Fuzzy objects

In this paper we deal with specific fuzzy sets, that represent spatial image objects and the imprecision attached to them. They are defined as follows.

Let us denote by $\mathscr{S}$ the space on which the image is defined (usually $\mathbb{R}^n$ or $\mathbb{Z}^n$). We denote by x, y, etc., the spatial variables, i.e. points of $\mathscr{S}$ (pixels or voxels).

We denote by $d_{\mathscr{S}}(x, y)$ the spatial distance between two points x and y of $\mathscr{S}$ (related to the Cartesian space they are belonging to and independent of their membership to any possible fuzzy set). Generally $d_{\mathscr{S}}$ is taken as the Euclidean distance on $\mathscr{S}$.

A crisp object is, as usual, a subset of $\mathscr{S}$. Similarly, a fuzzy object is defined as a fuzzy subset of $\mathscr{S}$. A fuzzy object is defined bi-univoquely by its membership function, denoted by Greek letters (μ, ν, etc.). A membership function characterizing a fuzzy object is therefore a function, say μ, from $\mathscr{S}$ into [0,1]. For each x in $\mathscr{S}$, $\mu(x)$ is a value in [0,1] which represents the membership degree of the point x to the fuzzy set μ. Such a representation allows for a direct representation of the spatial information. We denote by $\mathscr{F}$ the set of all fuzzy sets defined on $\mathscr{S}$.

For any two fuzzy objects μ and ν, we denote by $d(\mu, \nu)$ their distance. The definition of distances between fuzzy objects is the scope of this paper.

Since we are mainly interested here in the type of information that is included in the various distance definitions, we assume that the fuzzy sets satisfy the necessary properties such that all mathematical expressions are well defined. For instance in the continuous case, several definitions assume that the membership functions are Lebesgue integrable. This will not be specified in the following. Moreover, in most cases we will restrict to the discrete bounded case (i.e. membership functions defined on $\mathbb{Z}^n$ and having a bounded support), since this is the most useful case in image processing.

2.2. Types of problems related to distances

Several problems can be addressed when fuzzy distances are concerned. We distinguish three of them, briefly addressed yet in [15]:

- distances between two points in a fuzzy set,
- distances from a point to a fuzzy set,
- and distances between two fuzzy sets.

The first type of distance is the less treated in the literature. In the crisp case, this kind of distance is widely used in classical image processing and pattern recognition [16]. The definition of its fuzzy equivalent should lead to the design of new tools for generalizing classical methods when imprecision in structures and images has to be taken into account. We proposed in [17] to define a distance between two points in a fuzzy set as a fuzzy generalization of the concept of geodesic distance in a crisp set, by introducing fuzzy connectivity. Typical applications in fuzzy image processing consist in finding the best path in the geodesic sense in a spatial fuzzy set representing some objective function (satisfiability of a property, security areas around objects, etc.). Fuzzy geodesic distance is also the basis for fuzzy geodesic operators, e.g. morphological ones [18,19]. This type of distance is not considered in the following.

Distances from a point to a fuzzy set do not deserve much attention in the literature, although they are useful in image processing: they can be used for classification purposes where a point has to be attributed to the nearest fuzzy class, or when considering distance from a point to the complement of a fuzzy set μ, we obtain the basic information for computing a fuzzy skeleton of μ. We defined such distances based on fuzzy mathematical morphology in [20]. They are just mentioned in Section 4 since they may serve as a basis for defining distances between two fuzzy sets, but they are not further investigated here. We restrict this paper to the third kind of distances (between two fuzzy sets). It is the most widely addressed in the literature, but not often in the context of image processing. We think that the specificities of image information call for a study of the existing definitions in terms of image properties they include, and even for the definition of new ones. Applications of such distances cover a very large field, including image registration, assessment of relationships between image components, comparison of imprecise image objects, structural pattern recognition, etc. Roughly speaking, these applications can be grouped into two classes. The first class deals with distances dedicated to the comparison of shapes, these shapes being possibly contained in different images, or represent one image object and one model object. The concerned applications are related to registration and to recognition. The second class deals with distances between two objects in the same image, and provides measures for quantifying how far one object is from the other. It can also serve for model-based pattern recognition, as a relationships between image (respectively model) objects. For instance, if we consider a graph-based recognition method, where the objects of the scene are the modes of the scene, then distances of the first class provide a way to compare nodes in two graphs, while distances of the second class can be considered as attributes of the arcs between two nodes in each graph.

2.3. Properties of proximity relationships

Since the definitions summarized in this paper do not always satisfy strictly the properties of a distance (or metric), we should rather speak of more general proximity functions. However for sake of simplicity we will keep the term distance. The main classes of proximity measures are recalled in this section.

2.3.1. Definitions

A metric is a positive function d such that

(1) $\forall \mu \in \mathscr{F}$, $d(\mu,\mu)=0$ (reflexivity),
(2) $\forall (\mu,\nu) \in \mathscr{F}^2$, $d(\mu,\nu)=0 \Rightarrow \mu=\nu$ (separability),
(3) $\forall (\mu,\nu) \in \mathscr{F}^2$, $d(\mu,\nu)=d(\nu,\mu)$ (symmetry),
(4) $\forall (\mu,\nu,\xi) \in \mathscr{F}^3$, $d(\mu,\nu) \leqslant d(\mu,\xi)+d(\xi,\nu)$ (triangular inequality).

Several kinds of measures can be defined with less requirements: a pseudometric is a function satisfying 1, 3 and 4 (separability does not necessarily hold), a semi-metric satisfies 1, 2 and 3 (and not the triangular inequality), a semi-pseudometric satisfies only 1 and 3, etc. (see e.g. [21]).

Since distances may be derived from similarity measures, we recall here the definition of this concept.

A similarity relation [22] is a function s taking values in [0,1], such that

(1) $\forall \mu \in \mathscr{F}$, $s(\mu,\mu)=1$ (reflexivity),
(2) $\forall (\mu,\nu) \in \mathscr{F}^2$, $s(\mu,\nu)=s(\nu,\mu)$ (symmetry),
(3) $\forall (\mu,\nu,\xi) \in \mathscr{F}^3$, $t[s(\mu,\xi), s(\xi,\nu)] \leqslant s(\mu,\nu)$ (t-transitivity, where t is a t-norm).

A similarity relation is also called t-indistinguishability or t-equivalence.

If we set $d=1-s$, obviously d is a semi-pseudometric. If $t=\min$, then we also have

$$\forall (\mu,\nu,\xi) \in \mathscr{F}^3,\ d(\mu,\nu) \leqslant \max[d(\mu,\xi),\ d(\xi,\nu)]$$

which is a property of a hyper-metric. If t is the Lukasiewicz t-norm (i.e. $t(a,b)=\max(0,a+b-1)$), then d satisfies also the triangular inequality and is a pseudometric. If f is an additive generator (typically like the functions used for generating continuous Archimedian t-norms [23]), then $d=f \circ s$ is a pseudometric (taking values in $\mathbb{R}^+$) if and only if the t-norm generated by f is less than t [24]. A similar relationship holds between a metric and a t-equality (i.e. a similarity such that $s(\mu,\nu)=1$ if and only if $\mu=\nu$).

From a topological point of view, the definition of a metric d on $\mathscr{F}$ induces a topology on $\mathscr{F}$, and therefore a continuity. It has been studied for instance in [25] for the case of the fuzzy Hausdorff distance. Partial results can also be obtained if d has less properties: if we set $cl(\mu)=\{\nu \in \mathscr{F},\ d(\mu,\nu)=0\}$ for d being a semi-pseudometric, then the function cl is a pre-closure on $\mathscr{F}$, which therefore defines a pretopology on $\mathscr{F}$ (see e.g. [26,27]). Conversely, we may derive a semi-pseudometric from any (non-idempotent) adherence defined on $\mathscr{F}$.

2.3.2. Needs in image processing and pattern recognition

Although we may speak about distances between image objects in a very general way, this expression does not make necessarily the assumption that we are dealing with true metrics. For several applications in image processing, it is not sure that all properties are needed.

An important use of distances is related to the comparison of shapes, which reinforces the interest of deriving distances from similarities. The concept of similarities between objects, in particular image objects, contains some subjective aspects. As already stated by Poincaré at the beginning of the century, and underlined by several authors in the fuzzy sets domain (see e.g. [28,29]), subjective similarities does not require to be transitive. This induces a loss of triangular inequality in the derived distance. In image processing, typically for applications where image objects have to be compared to models, the triangular inequality is of no use, since the two arguments of the distance function belong to two different sets of objects. For such applications, semi-metrics or even semi-pseudometrics may be sufficient.

We may even go farther in this direction. Indeed, since a semi-pseudometric does not satisfy the separability property, the study of the equation $d(\mu,\nu)=0$ can be exploited in terms of pattern recognition. For instance if we build classes according to prototypes, this equation can be used as a classification rule: every object which is indistinguishable from a prototype will be added to the corresponding class. This has been developed in the context of pre-topologies [26,30]. It is the non-idempotency of the adherence function in a pre-topology that allows to aggregate objects to a class. This is again an argument in favor of semi-pseudometrics.

Another aspect that can be useful in image processing and pattern recognition is the link existing between semi-metrics and fuzzy partitions derived from a t-indistinguishability relation. This clearly finds applications as soon as the recognition or classification problem can be stated as the (fuzzy) partitioning of the set of objects.

2.4. Representation of fuzzy distances

In the previous sections, we always assumed that d takes values in $\mathbb{R}^+$ (or more specifically in [0,1] for some of them). This corresponds to the most used representation of the distance between two fuzzy sets, as a number. However, since we consider fuzzy sets, i.e. objects that are imprecisely defined, we may expect that the distance between them is imprecise too. This argument is advocated in particular in [31,32]. Then the distance is better represented as a fuzzy set, and more precisely as a fuzzy number (a convex upper semi-continuous fuzzy set on $\mathbb{R}^+$ having a bounded support).

In [32], Rosenfeld defines two concepts that will be used in the sequel. One is distance density, denoted by $\delta(\mu, v)$, and the other distance distribution, denoted by $\Delta(\mu, v)$, both being fuzzy sets on $\mathbb{R}^+$. They are linked together by the following relation:

$$\Delta(\mu, v)(n) = \int_0^n \delta(\mu, v)(n')\,\mathrm{d}n'. \tag{1}$$

While the distance distribution value $\Delta(\mu, v)(n)$ represents the degree to which the distance between μ and v is less than n, the distance density value $\delta(\mu, v)(n)$ represents the degree to which the distance is equal to n.

Finally, the concept of distance can be represented as a linguistic variable. This assumes a granulation [33] of the set of possible distance values into symbolic classes such as "near", "far", etc., each of these classes being defined as a fuzzy set. This approach has been drawn e.g. in [34–36].

2.5. Overview of the main approaches

In this section, we briefly summarize the main approaches that can be followed in order to define a fuzzy distance. These include:

- approaches that rely on the definition of a crisp distance and try to generalize them,
- approaches that infer a distance from a similarity function,
- approaches that deduce a distance from set relationships between both sets (or other types of relationships),
- symbolic approaches.

2.5.1. Generalizing a crisp distance to a fuzzy one

In this section, we consider the class of approaches to define a fuzzy distance that rely on extension of a given crisp distance. They belong to the general problem of extending a relationship R_B between two binary objects to its fuzzy equivalent R (fuzzy relationship between two fuzzy objects). Instantiations of the described methods to the case of distance are provided in Sections 3 and 4.

From α-cuts: A way to define crisp sets from a fuzzy set consists in taking the α-cuts of this set. Therefore, one class of methods relies on the application of the relationship R_B to each α-cut. This gives rise to two different "fuzzification" methods in the literature.

The first fuzzification method consists in "stacking" the results obtained with binary operations on the α-cuts: the fuzzy equivalent R of R_B is defined as (see e.g. [8,35,37]):

$$R(\mu, v) = \int_0^1 R_B(\mu_\alpha, v_\alpha)\,\mathrm{d}\alpha, \tag{2}$$

where μ_α denotes the α-cut of μ, or by a double integration as

$$R(\mu, v) = \int_0^1 \int_0^1 R_B(\mu_\alpha, v_\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{3}$$

Other fuzzification equations are possible, like

$$R(\mu,v) = \sup_{\alpha\in[0,1]} \min(\alpha, R_B(\mu_\alpha, v_\alpha)) \quad \text{or}$$

$$R(\mu, v) = \sup_{\alpha\in[0,\ 1]} (\alpha R_B(\mu_\alpha, v_\alpha)), \tag{4}$$

the first one of these equations being meaningful if R_B takes values in [0,1].

This approach has been applied to the definition of several fuzzy operations, for instance connectivity [6], fuzzy mathematical morphology [8], fuzzy adjacency [10], and of course distances [15,20,37] as will be seen later.

The second fuzzification method is the extension principle [38], which leads in the general case to a fuzzy number (rather than a crisp number):

$$\forall n \in \mathscr{V}(R_B), R(\mu, v)(n) = \sup_{R_B(\mu_\alpha, v_\alpha) = n} \alpha, \tag{5}$$

where $\mathscr{V}(R_B)$ denotes the image of R_B, i.e. the set of values taken by R_B ($\mathbb{R}^+$ or [0,1] in the case of distances).

Translating binary equations into fuzzy ones: Another way to proceed, in order to derive a fuzzy definition from a crisp one, consists in translating binary equations into their fuzzy equivalent: intersection is replaced by a t-norm, union by a t-conorm, sets by membership functions, etc. Examples can be found for defining fuzzy morphology [8], fuzzy inclusion [9], etc.

This translation is particularly straightforward if the binary relationship can be expressed in set theoretical and logical terms. This can be obtained in a natural way for several distances, like nearest point distance or Hausdorff distance [20]. This remark endows methods based on mathematical morphology with a particular interest, since mathematical morphology is mainly based on set theory. This approach will be used in Section 4.

2.5.2. Distances from similarity

We already mentioned that a distance can be derived formally from a similarity measure (see for instance [22,24,39–42]). Then the problem amounts to define the similarity measure. This can be addressed using one of the previous methods, given a similarity between crisp sets. However, because of the links between similarity and pattern recognition problems, this approach is often used for comparing objects based on some features, possibly fuzzy ones, that are extracted from the images in preliminary stages. Then the similarity concerns these features, and not the objects as spatial fuzzy sets. This may explain why this approach leads mainly to distances dealing with membership functions only (Section 3).

Similarity-based approaches can benefit from the existing algorithms for checking if a relation is a similarity, in particular if it satisfies the transitivity property (e.g. [43,44]).

2.5.3. Distances from set relationships

Set relationships provide a lot of information for the comparison of objects, typically in the case where image objects have to be compared with some models or prototypes. Similar objects are expected to strongly overlap and to have reduced differences. We have chosen to present here the approach proposed in [45,46], where a very useful typology of comparison measures is proposed.

In this work, a comparison measure is generally defined as a function of three variables $F_S[M(\mu \cap \nu), M(\nu - \mu), M(\mu - \nu)]$, where M is a fuzzy set measure (e.g. fuzzy cardinality) and $-$ denotes a difference operator (such that $\mu \subset \nu \Rightarrow \mu - \nu = \emptyset$, and $\mu \subset \mu' \Rightarrow \mu - \nu \subset \mu' - \nu$). This approach is closed from Tversky definitions [47]. Then specific types of comparison measures are defined:

- a similitude measure is a comparison measure such that $F_S(x, y, z)$ is non-decreasing with respect to x and non-increasing with respect to y and z (this corresponds to the fact that two fuzzy sets are more similar if they have a greater intersection and less difference);
- a satisfiability measure is a similitude measure such that $F_S(0, y, z) = 0$, $F_S(x, 0, z) = 1$, and which does not depend on z (this corresponds to the case where the first object is considered as a reference to which the other is compared);
- an inclusion measure is a reflexive similitude measure such that $F_S(0, y, z) = 0$ and F_S does not depend on z;
- a resemblance measure is a symmetrical and reflexive measure;
- a dissimilarity measure is a comparison measure taking value 0 if $\mu = \nu$, and such that F_S is independent of x and increasing with respect to y and z.

A distance between two fuzzy sets can be derived from a dissimilarity measure, or from $1 - F_S$ if F_S defines a similitude measure. Several distances that have been proposed in the literature can be classified from this point of view.

2.5.4. Distances from other relationships

When distances are mainly used for comparing shapes, they may be derived from other relationships between objects, not only metric ones. Set relationships can be used as shown in the previous section, but also several other ones, like geometrical features extracted from the object or any other type of attribute, and topological relationships like "overlap", "meet", etc. [48–50]. Since such measures do not necessarily include information on the spatial distance, they are mainly found in the first class of definitions (Section 3) and used for model-based pattern recognition, for approaches relying on prototypes, for applications like indexing and searching in image databases (as in [50,51] for instance).

Such methods are often related to similarity-based measures.

2.5.5. Symbolic approaches

We mean by "symbolic approaches" methods that try to define linguistic variables representing distances (the last type of representation mentioned before). In image processing, the problem amounts to derive symbolic representations from the numerical information carried by the image and from computation on it (see e.g. [35]). These representations then provide a kind of summarization of the image content related to metric information. These approaches are not further detailed here, and the following sections are restricted to numerical approaches, where distances are evaluated as numbers or fuzzy numbers.

3. Distances between two fuzzy sets by comparing membership functions

In this section, we review the main distances proposed in the literature that aim at comparing membership functions. They have generally been proposed in a general fuzzy set framework, and not specifically in the context of image processing. They do not really include information about spatial distances. The classification chosen here is inspired from the one found in [14]. Similar classifications can be found in [52–54].

3.1. Functional approach

The functional approach is probably the most popular. It relies on a L_p norm between μ and ν, leading to the following generic definition [21,31,55]:

$$d_p(\mu,\nu) = \left[\int_{x\in\mathscr{S}} |\mu(x) - \nu(x)|^p \right]^{1/p}, \tag{6}$$

$$d_\infty(\mu,\nu) = \sup_{x\in\mathscr{S}} |\mu(x) - \nu(x)|. \tag{7}$$

d_p is a pseudometric, while d_∞ is a metric. In general, d_p does not converge towards d_∞ when p becomes infinite, but it converges towards [21]:

$$d_{\text{Ess Sup}}(\mu,\nu) = \inf\{k\in\mathbb{R}, \lambda(\{x, |\mu(x) - \nu(x)| > k\}) = 0\}, \tag{8}$$

where λ denotes the Lebesgue measures on $\mathscr{S}$. $d_{\text{Ess Sup}}$ is a pseudometric, called essential supremum, and related to d_∞ by the relation $d_{\text{Ess Sup}} \leqslant d_\infty$. Equality does not hold in the general continuous case (a counterexample can be found in [21]).

In the discrete finite case, these definitions become

$$d_p(\mu,\nu) = \left[\sum_{x\in\mathscr{S}} |\mu(x) - \nu(x)|^p \right]^{1/p}, \tag{9}$$

$$d_\infty(\mu,\nu) = \max_{x\in\mathscr{S}} |\mu(x) - \nu(x)|. \tag{10}$$

In this case, they are all metrics. Therefore, this approach is also called metric-based in [54].

A noticeable property of d_p is that it takes a constant value if the supports of μ and ν are disjoint. In such cases, we have

$$d_p(\mu,\nu) = |\mu| + |\nu|, \tag{11}$$

where $|\mu|$ denotes the fuzzy cardinality of μ, and for d_∞ we have

$$d_\infty(\mu,\nu) = \max\left[\sup_{x\in\mathscr{S}} \mu(x), \sup_{x\in\mathscr{S}} \nu(x) \right], \tag{12}$$

which is equal to 1 if the fuzzy sets are normalized.

These equations show that, as soon as the support of μ and ν are disjoint, the value taken by their distance is constant, irrespective of how far the supports are from each other in $\mathscr{S}$.

A slightly different version of d_1 has been proposed in [52,56], where the distance is normalized by $|\mathscr{S}|$ (cardinality of $\mathscr{S}$). However, this normalization does not change the properties, neither the type of information taken into account. It allows an easier link to similarity.

The distance d_∞ is also called geometrical distance in [52]. However, this definition (as well as the general definition d_p) considers only the geometry of the two fuzzy sets with respect to each other, in terms of shape of the membership function, but does not include the geometry related to $d_{\mathscr{S}}$.

The distance d_1 has been used in a pyramidal approach in image processing in [40] for recognizing objects based on their attributes. In this example, the fuzzy sets do not represent the objects themselves but fuzzy attributes of the objects. Therefore, the spatial information is not taken into account at the level of the distance formulation but is rather included implicitly in the type of features used.

Other forms of distances can be found in this class. For instance, in [53], the following form is proposed (in the finite discrete case):

$$d(\mu,\nu) = \frac{\sum_{x\in\mathscr{S}} |\mu(x) - \nu(x)|}{\sum_{x\in\mathscr{S}} (\mu(x) + \nu(x))}. \tag{13}$$

This equation corresponds to a normalization of d_1 by the sum of the cardinality of μ and ν. Again, its value is constant if the supports of both fuzzy sets are disjoint, the constant being equal to 1.

3.2. Information theoretic approach

Based on their definition of fuzzy entropy $E(\mu)$, de Luca and Termini define a pseudometric as [57]

$$d(\mu,\nu) = |E(\mu) - E(\nu)|, \tag{14}$$

with

$$E(\mu) = -K \sum_{x\in\mathscr{S}} [\mu(x)\log\mu(x) + (1-\mu(x))\log(1-\mu(x))], \tag{15}$$

K being a normalization constant.

This distance does not satisfy the separability condition. This can be overcome by considering the quotient space obtained through the equivalence relation $\mu \sim \nu \Leftrightarrow E(\mu) = E(\nu)$. However this is not suitable for image processing. Indeed, since the entropy of a crisp set is zero, two crisp structures in an image belong to the same equivalence class, even if they are completely different. One main drawback of this approach is that the distance is based on the comparison of two global measures performed on μ and ν separately: there are no linking points of μ to points of ν, which is of reduced interest for computing distances.

Entropy functions under similarity [58,59] combine this approach with the membership comparison approach. It has been applied in decision problems (in particular for questionnaires) but to our knowledge not in image processing.

Based on a similar approach, a notion of fuzzy divergence (which can be interpreted as a distance) has been introduced in [60], by mimicking Kulback's approach [61]:

$$d(\mu, \nu) = \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} [D_x(\mu, \nu) + D_x(\nu, \mu)] \quad (16)$$

with

$$D_x(\mu, \nu) = \mu(x) \log \frac{\mu(x)}{\nu(x)} + (1 - \mu(x)) \log \frac{1 - \mu(x)}{1 - \nu(x)}$$

and the convention $0/0 = 1$. This distance is positive, symmetrical, but does not satisfy the triangular inequality. Morever, it is always equal to 0 for crisp sets.

3.3. Set theoretic approach

In this approach, distance between two fuzzy sets is seen as a set dissimilarity function, based on fuzzy union and intersection. Examples are given in [14]. The basic idea is that the distance should be larger if the two fuzzy sets weakly intersect. Most of the proposed measures are inspired from the work by Tversky [47] that proposes two parametric similarity measures between two sets A and B:

$$\theta f(A \cap B) - \alpha f(A - B) - \beta f(B - A) \quad (17)$$

and in a rational form

$$\frac{f(A \cap B)}{f(A \cap B) + \alpha f(A \cap \bar{B}) - \beta f(B \cap \bar{A})}, \quad (18)$$

where $f(X)$ is typically the cardinality of X, α, β and θ are parameters leading to different kinds of measures, and $\bar{B}$ denotes the complement of B.

Let us mention a few examples (they are given in the finite discrete case). A measure being derived from the second Tversky measure by setting $\alpha = \beta = 1$ has been used by several authors [14,39,52,53,54,62,63]:

$$d(\mu, \nu) = 1 - \frac{\sum_{x \in \mathcal{S}} \min[\mu(x), \nu(x)]}{\sum_{x \in \mathcal{S}} \max[\mu(x), \nu(x)]}. \quad (19)$$

This distance is a semi-metric, and always takes the constant value 1 as soon as the two fuzzy sets have disjoint supports. It also corresponds to the Jaccard index [63]. With respect to the typology presented in [46], this distance is a comparison measure, and more precisely a dissimilarity measure. Moreover, $1 - d$ is a resemblance measure. Applications in image processing can be found e.g. in [64], where it is used on fuzzy sets representing objects features (and not directly spatial image objects) for structural pattern recognition on polygonal 2D objects.

A slightly different formula has been proposed in [56], which however translates a similar idea:

$$d(\mu, \nu) = 1 - \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} \frac{\min[\mu(x), \nu(x)]}{\max[\mu(x), \nu(x)]} \quad (20)$$

with the convention $0/0 = 1$. It is a semi-metric. It takes the constant value 1 if the two fuzzy sets have disjoint supports, without any other condition on their relative position in the space.

Another measure takes into account only the intersection of the two fuzzy sets [14,52,54]:

$$d(\mu, \nu) = 1 - \max_{x \in \mathcal{S}} \min[\mu(x), \nu(x)]. \quad (21)$$

It is a semi-pseudometric if the fuzzy sets are normalized. Again it is a dissimilarity measure, and $1 - d$ is a resemblance measure. It is always equal to 1 if the supports of μ and ν are disjoint.

If we set $(\mu \,\Box\, \nu)(x) = \max[\min(\mu(x), 1 - \nu(x)), \min(1 - \mu(x), \nu(x))]$, two other distances can be derived, as [54,14]

$$d(\mu, \nu) = \sup_{x \in \mathcal{S}} (\mu \,\Box\, \nu)(x), \quad (22)$$

$$d(\mu, \nu) = \sum_{x \in \mathcal{S}} (\mu \,\Box\, \nu)(x). \quad (23)$$

These two distances are symmetrical measures. They are separable only for binary sets. Also we have $d(\mu, \mu) = 0$ only for binary sets. They are dissimilarity measures. The first one is equal to 1 if μ and ν have disjoint supports and are normalized (if they are not normalized, then this constant value is equal to the maximum membership value of μ and ν). The second measure is always equal to $|\mu| + |\nu|$ is μ and ν have disjoint supports.

These measures actually rely on measures of inclusion of each fuzzy sets in the other. Indeed, an inclusion index can be defined as [8,9] as

$$\mathcal{I}(\mu, \nu) = \inf_{x \in \mathcal{S}} T[\mu(x), 1 - \nu(x)], \quad (24)$$

where T is a t-conorm. Since the distance should be small if the two sets have a small degree of equality (the equality between μ and ν can be expressed by "μ included in ν and ν included in μ", which leads to an easy transposition to fuzzy equality), a distance may be defined from an inclusion degree as

$$d(\mu, \nu) = 1 - \min[\mathcal{I}(\mu, \nu), \mathcal{I}(\nu, \mu)]. \quad (25)$$

By taking $T = \max$, we recover the definition derived from $(\mu \,\Box\, \nu)$. This approach has been used in [39,65]. Other choices of T may lead to different properties of d. For instance, if T is taken as the Lukasiewicz t-conorm (bounded sum), then $(\mu \,\Box\, \nu)(x) = |\mu(x) - \nu(x)|$. Therefore we have:

$$\sup_{x \in \mathcal{S}} (\mu \,\Box\, \nu)(x) = d_\infty(\mu, \nu), \quad (26)$$

and

$$\sum_{x \in \mathscr{S}} (\mu \,\square\, \nu)(x) = d_1(\mu,\nu). \tag{27}$$

In this case, both distances are metrics in the discrete finite case.

These measures have been applied in image processing for image databases applications in [54].

Other inclusion indexes can be defined, e.g. from Tversky measure by setting $\alpha = 1$ and $\beta = 0$, leading to $f(A \cap B)/f(A)$ [63].

The last definitions given by Eqs. (21) and (22) are, respectively, equivalent to $1 - \Pi(\mu;\nu)$ and $1 - \max[N(\mu;\nu), N(\nu;\mu)]$ (where Π and N are possibility and necessity functions) used in fuzzy pattern matching [66,67], which has a large application domain, including image processing (see e.g. [68]). It is interesting to note that they are related to fuzzy mathematical morphology, since $\Pi(\mu;\nu)$ corresponds to the dilation of μ by ν at origin, while $N(\mu;\nu)$ corresponds to the erosion of μ by ν at origin. These definitions can be straightforwardly generalized to fuzzy union and intersection derived from t-norms and t-conorms, leading to a correspondence with other forms of fuzzy mathematical morphology [8].

Such generalizations using t-norm and t-conorm for set relationships can be done for all definitions presented in this section.

3.4. Pattern recognition approach

This approach consists in first expressing each fuzzy set in a feature space (for instance, cardinality, moments, skewness) and to compute the Euclidean distance between two feature vectors [14] or attribute vectors [51]. This approach may take advantage of some of the previous approaches, for instance, by using entropy or similarity in the set of features. It has been applied, for instance, for database applications [51].

A similar approach, called signal detection theory, has been proposed in [54]. It is based on counting the number of similar and different features.

A particular form of distances between attributes can be found in [52], where the distance is defined from vectorial representations a and b as

$$1 - \frac{a \cdot b}{\max(a \cdot a, b \cdot b)}. \tag{28}$$

This form is very close to correlation-based approaches, such as the one described in [56]

$$d(\mu,\nu) = 1 - \frac{\sum_{x\in\mathscr{S}}[\mu(x)\nu(x) + (1-\mu(x))(1-\nu(x))]}{\sqrt{\sum_{x\in\mathscr{S}}[\mu(x)^2 + (1-\mu(x))^2]\sum_{x\in\mathscr{S}}[\nu(x)^2 + (1-\nu(x))^2]}}. \tag{29}$$

The Bhattacharya distance [62] can also be attached to this class. It is defined as

$$d(\mu,\nu) = \left[1 - \int_{\mathscr{S}} \sqrt{\frac{\mu(x)}{|\mu|}\frac{\nu(x)}{|\nu|}}\,\mathrm{d}x\right]^{1/2}. \tag{30}$$

It has been used in image processing for classification in satellite images in [69].

4. Combination of spatial and membership comparisons

The second class of methods tries to include the spatial distance $d_{\mathscr{S}}$ in the distance between μ and ν. In contrary to the definitions given in Section 3, in this second class the membership values at different points of $\mathscr{S}$ are linked using some formal computation, making the introduction of $d_{\mathscr{S}}$ possible. This leads to definitions that do not share the drawbacks of previous approaches, for instance when the supports of the two fuzzy sets are disjoint.

4.1. Geometrical approach

The geometrical approach consists in generalizing one of the distances between crisp sets. This has been done, for instance, for nearest point distance [31,32], mean distance [32], Hausdorff distance [31], and could easily be extended to other distances (see e.g. [70] for a review of crisp set distances). These generalizations follow four main principles.

The first one consists in considering fuzzy sets in a n-dimensional space as $(n+1)$-dimensional crisp sets and then in using classical distances [71]. However, this is often not satisfactory in image processing because the n dimensions of $\mathscr{S}$ and the membership dimension (values in [0,1]) have completely different interpretations, and treating them in a unique way is questionable.

The second principle is a fuzzification principle (see Section 2.5): let D be a distance between crisp sets, then its fuzzy equivalent is defined by

$$d(\mu,\nu) = \int_0^1 D(\mu_\alpha, \nu_\alpha)\,\mathrm{d}\alpha \tag{31}$$

or by a discrete sum if the fuzzy membership functions are piecewise constant [14,37] (μ_α denotes the α-cut of μ). In this way, $d(\mu,\nu)$ inherits the properties of the chosen crisp distance. Another way to consider the fuzzification principle consists in using a double integration (see Section 2.5). However using this double fuzzification, some properties of the underlying distance may be lost.

The third principle consists in weighting distances by membership values. For the mean distance this leads, for instance, to [32]

$$d(\mu,\nu) = \frac{\sum_{x\in\mathscr{S}}\sum_{y\in\mathscr{S}} \mathrm{d}_{\mathscr{S}}(x,y)\min[\mu(x),\nu(y)]}{\sum_{x\in\mathscr{S}}\sum_{y\in\mathscr{S}}\min[\mu(x),\nu(y)]}. \tag{32}$$

The last approach consists in defining a fuzzy distance as a fuzzy set on $\mathbb{R}^+$ instead of as a crisp number using the extension principle (see Section 2.5). For the nearest point distance this leads to [32]

$$d(\mu, \nu)(r) = \sup_{x,y,d_{\mathscr{S}}(x,y) \leq r} \min[\mu(x), \nu(y)]. \tag{33}$$

The Hausdorff distance is probably the distance between sets, the fuzzy extension of which has been the most widely studied. One reason for this may be that it is a true metric in the crisp case, while other set distances like minimum or average distances have weaker properties. Another reason is that it has been used to determine a degree of similarity between two objects, or between an object and a model [72]. Extensions of this distance have been defined using fuzzification over the α-cuts and using the extension principle [14,25,73,74,75,76]. Other authors use the Hausdorff distance between the endographs of the two membership functions [25]. Several generalizations of Hausdorff distance have also been proposed under the form of fuzzy numbers [31]. Extensions of the Hausdorff distance based on fuzzy mathematical morphology have also been developed [20] and are presented in the next section.

Extensions of these definitions may be obtained by using other weighting functions, for instance, by using t-norms instead of min.

These distances share most of the advantages and drawbacks of the underlying crisp distance [70]: computation cost can be high (it is already high for several crisp distances); moreover, interpretation and robustness strongly depend on the chosen distance (for instance, Hausdorff distance is noise sensitive, whereas mean distance is not).

4.2. Morphological approach

We proposed in [20] original approaches for defining fuzzy distances taking into account spatial information, which are based on fuzzy mathematical morphology. They are summarized below.

4.2.1. Distances from a point to a fuzzy set

Distances from a point to a fuzzy set can be defined using a weighting approach or using a fuzzification from α-cuts. In this way, they are defined as numbers.

We propose an original approach for defining the distance $d(x, \mu)$ from a point x of $\mathscr{S}$ to a fuzzy object μ as a fuzzy number, by translating crisp equations into their fuzzy equivalent. Standard expressions for the distance involve concepts that are not set theoretical ones, and are therefore not trivial to translate. Therefore, the easiest way to perform this translation is to find a formalism where distances are expressed in set theoretical terms. This formalism is provided by mathematical morphology, since the distance from a point to a set can be expressed in terms of morphological dilation, as well as several distances between two sets. The translation of dilation in fuzzy terms can be achieved with good properties using the framework of fuzzy mathematical morphology we developed in [8].

In the crisp case, and in a finite discrete space, we have respectively for $n = 0$ and for $n > 0$:

$$\mathrm{d}_B(x, X) = 0 \Leftrightarrow x \in X, \tag{34}$$

$$\mathrm{d}_B(x, X) = n \Leftrightarrow x \in D^n(X) \quad \text{and} \quad x \notin D^{n-1}(X), \tag{35}$$

where D^n denotes the dilation by a ball of radius n centered at the origin of $\mathscr{S}$ (and $D^0(X) = X$) (see e.g. [77] for a study of discrete balls and discrete distances in the crisp case). In this case, the extensivity property of the dilation holds [16], and $x \notin D^{n-1}(X)$ is equivalent to $\forall n' < n$, $x \notin X^{n'}(X)$. Eq. (35) is equivalent to

$$x \in D^n(X) \cap [D^{n-1}(X)]^C, \tag{36}$$

where A^C denotes the complement set of A in $\mathscr{S}$. This is a pure set theoretical expression, that we can now translate into fuzzy terms. This leads to the following definition of the degree to which $d(x, \mu)$ is equal to n:

$$\delta_{(x,\mu)}(0) = \mu(x), \tag{37}$$

$$\delta_{(x,\mu)}(n) = t[D^n_\nu(\mu)(x),\, c[D^{n-1}_\nu(\mu)(x)]], \tag{38}$$

where t is a t-norm (fuzzy intersection), c a fuzzy complementation (typically $c(a) = 1 - a$ for $a \in [0,1]$), and ν a fuzzy structuring element used for performing the dilation. Several choices of ν are possible. It can be simply the unit ball, or a fuzzy set representing, for instance, the smallest sensitive unit in the image, along with the imprecision attached to it. In this case, ν has to be equal to 1 at the origin of $\mathscr{S}$, such that the extensivity of the dilation still holds [8].

The properties of this definition are the following [20]: If $\mu(x) = 1$, $\delta_{(x,\mu)}(0) = 1$ and $\forall n > 0$, $\delta_{(x,\mu)}(n) = 0$, i.e. the distance is a crisp number in this case. If μ and ν are binary, the proposed definition coincides with the binary one. The fuzzy set $\delta_{(x,\mu)}$ can be interpreted as a density distance, from which a distance distribution can be deduced by integration. Finally, $\delta_{(x,\mu)}$ is a non-normalized fuzzy number (in the discrete finite case).

From this definition, distances between two fuzzy sets can be derived using supremum or infimum computation of fuzzy numbers using the extension principle [62]. The details are given in [20].

4.2.2. Distances between two fuzzy sets

We defined distances between two fuzzy objects using a morphological approach in [20], in an original way. They are obtained by direct translation of crisp equations expressing distances in terms of mathematical morphology into fuzzy ones (see Section 2.5). We just give the examples of nearest point distance and Hausdorff distance.

In the binary case, for $n > 0$, the nearest point distance can be expressed in morphological terms as

$$\mathrm{d}_N(X,Y) = n \Leftrightarrow D^n(X) \cap Y \neq \emptyset \quad \text{and} \quad D^{n-1}(X) \cap Y = \emptyset \tag{39}$$

and the symmetrical expression. For $n = 0$ we have

$$\mathrm{d}_N(X,Y) = 0 \Leftrightarrow X \cap Y \neq \emptyset. \tag{40}$$

The translation of these equivalences provides, for $n > 0$, the following distance density:

$$\delta_N(\mu,\mu')(n) = t\left[\sup_{x\in\mathcal{S}} t[\mu'(x), D^n_\nu(\mu)(x)],\; c\left[\sup_{x\in\mathcal{S}} t[\mu'(x), D^{n-1}_\nu(\mu)(x)]\right]\right] \tag{41}$$

or a symmetrical expression derived from this one, and

$$\delta_N(\mu,\mu')(0) = \sup_{x\in\mathcal{S}} t[\mu(x),\mu'(x)]. \tag{42}$$

This expression shows how the membership values to μ' are included, without involving the extension principle. Like for the nearest point distance, we can extend the Hausdorff distance by translating directly the binary equation defining the Hausdorff distance:

$$d_H(X,Y) = \max\left[\sup_{x\in X} d_B(x,Y), \sup_{y\in Y} d_B(y,X)\right]. \tag{43}$$

This distance can be expressed in morphological terms as

$$d_H(X,Y) = \inf\{n, X \subset D^n(Y) \text{ and } Y \subset D^n(X)\}. \tag{44}$$

From Eq. (44), a distance distribution can be defined, by introducing fuzzy dilation

$$\Delta_H(\mu,\mu')(n) = t\left[\inf_{x\in\mathcal{S}} T[D^n_\nu(\mu)(x), c(\mu'(x))],\; \inf_{x\in\mathcal{S}} T[D^n_\nu(\mu')(x), c(\mu(x))]\right], \tag{45}$$

where c is a complementation, t a t-norm and T a t-conorm. A distance density can be derived implicitly from this distance distribution.

A direct definition of a distance density can be obtained from

$$d_H(X,Y) = 0 \Leftrightarrow X = Y \tag{46}$$

and for $n > 0$:

$$d_H(X,Y) = n \Leftrightarrow X \subset D^n(Y) \quad \text{and} \quad Y \subset D^n(X)$$

and

$$(X \not\subset D^{n-1}(Y) \text{ or } Y \not\subset D^{n-1}(X)). \tag{47}$$

Translating these equations leads to a definition of the Hausdorff distance between two fuzzy sets μ and μ' as a fuzzy number:

$$\delta_H(\mu,\mu')(0) = t\left[\inf_{x\in\mathcal{S}} T[\mu(x), c(\mu'(x))],\; \inf_{x\in\mathcal{S}} T[\mu'(x), c(\mu(x))]\right], \tag{48}$$

$$\delta_H(\mu,\mu')(n) = t\Bigg[\inf_{x\in\mathcal{S}} T[D^n_\nu(\mu)(x), c(\mu'(x))],\; \inf_{x\in\mathcal{S}} T[D^n_\nu(\mu')(x), c(\mu(x))],\; T\Big(\sup_{x\in\mathcal{S}} t[\mu(x), c(D^{n-1}_\nu(\mu')(x))],\; \sup_{x\in\mathcal{S}} t[\mu'(x), c(D^{n-1}_\nu(\mu)(x))]\Big)\Bigg]. \tag{49}$$

The above definitions of fuzzy nearest point and Hausdorff distances (defined as fuzzy numbers) between two fuzzy sets do not necessarily share the same properties as their crisp equivalent. This is due in particular to the fact that, depending on the choice of the involved t-norms and t-conorms, excluded-middle and non-contradiction laws may not be satisfied. All distances are positive, in the sense that the defined fuzzy numbers have always a support included in $\mathbb{R}^+$. By construction, all defined distances are symmetrical with respect to μ and μ'. The separability property is not always satisfied. However, if μ is normalized, we have for the nearest point distance $\delta_N(\mu,\mu)(0) = 1$ and $\delta_N(\mu,\mu)(n) = 0$ for $n > 1$. For the Hausdorff distance, $\delta_H(\mu,\mu')(0) = 1$ implies $\mu = \mu'$ for T being the bounded sum ($T(a,b) = \min(1, a + b)$), while it implies μ and μ' crisp and equal for $T = \max$. Also the triangular inequality is not satisfied in general.

4.3. Tolerance-based approach

This approach has been developed in [21]. The basic idea is to combine spatial information and membership values by assuming a tolerance value τ, indicating the differences that can occur without saying that the objects are no more similar. The proposed definitions are semi-pseudometrics and are derived from the functional approach (see Section 3). The authors first define a local difference between μ and ν at a point x of $\mathcal{S}$ as

$$d^\tau_x(\mu,\nu) = \inf_{y,z\in B(x,\tau)} |\mu(y) - \nu(z)|, \tag{50}$$

where $B(x,\tau)$ denotes the (spatial) closed ball centered at x of radius τ.

Then the functions d_p, d_∞ and $d_{\mathrm{Ess\,Sup}}$ are defined up to a tolerance τ as

$$d_p^\tau(\mu, \nu) = \left[\int_{\mathscr{S}} [d_x^\tau(\mu, \nu)]^p \, \mathrm{d}x \right]^{1/p}, \tag{51}$$

$$d_\infty^\tau(\mu, \nu) = \sup_{x \in \mathscr{S}} d_x^\tau(\mu, \nu), \tag{52}$$

$$d_{\mathrm{Ess\,Sup}}^\tau(\mu, \nu) = \inf\{k \in \mathbb{R}, \lambda(\{x \in \mathscr{S}, d_x^\tau(\mu, \nu) > k\}) = 0\}. \tag{53}$$

Several results are proved in [21], in particular about convergence: $d_p^\tau(\mu, \nu)$ converges towards $d_{\mathrm{Ess\,Sup}}^\tau(\mu, \nu)$ when p goes to infinity, all pseudometrics are decreasing with respect to τ, and converge towards d_p, d_∞ and $d_{\mathrm{Ess\,Sup}}$ when τ becomes infinitely small, for continuous fuzzy sets.

This approach has been extended in [27], by allowing the neighborhood around each point to depend on the point.

Note that this approach has strong links with morphological approaches, since the neighborhood considered around each point can be considered as a structuring element.

This approach has been illustrated on an example of noisy character recognition.

4.4. Graph theoretic approach

A similarity function between fuzzy graphs may also induce a distance between fuzzy sets. This approach contrasts with the previous ones, since the objects are no more represented directly as fuzzy sets on $\mathscr{S}$ or as vectors of attributes, but as higher level structures. Fuzzy graphs in image processing can be used for representing objects, as in [79], or a scene, as in [80]. In the first case, nodes are parts of the objects and arcs are links between these parts. In the example presented in [79] for character recognition, nodes are fuzzy sets representing features of a character, extracted by some image processing. In the second case, nodes are objects of the scene and arcs are relationships between these objects. In the example of [80], the nodes represent clouds extracted from satellite images. These two examples use different ways to consider distances (or similarity) between fuzzy graphs.

In [79], the distance is defined from a similarity between nodes and between arcs (both being fuzzy sets), given a correspondence between nodes (respectively between arcs). The similarity used compares only membership functions, using a set theoretic approach (see Section 3) and corresponds to Eq. (19). Although it has not been considered in this reference, spatial distance can then be taken into account if we include it in the attribute set. This idea is probably worth to be further developed.

Another way to consider distances between objects is in terms of cost of deformations to bring one set in correspondence with the other. Such approaches are particularly powerful in graph-based methods. The distance can then be expressed as the cost of the matching of two graphs, as done in [80] for image processing applications, or as the Levensthein distance accounting for the necessary transformations (insertions, substitutions, deletions) for going from the structural representation of one shape to the representation of the other [81]. In [80], the fuzzy aspect is taken into account as weighting factors, therefore the method is quite close of the weighted Levensthein distance of [81]. Spatial distances could also be introduced as one of the relationships between objects in these approaches.

A distance between conceptual graphs is defined in [82], as an interval $[N, \Pi]$ where N represents the necessity and Π the possibility, obtained by a fuzzy pattern matching approach. Although the application is not related to image processing, the idea of expressing similarity as an interval is interesting and could certainly be exploited in other domains. A second interest of this approach is that the nodes of the graph are concepts, which could be (although not explicitly mentioned in this reference) represented as fuzzy sets (like linguistic variables).

Although these examples are still far from the main concern of this paper, it is worth mentioning them, since they bring an interesting structural aspect that could be further developed.

4.5. Distance between two sets of fuzzy sets

An interesting generalization of previous approaches would be to compute a distance between two sets of fuzzy sets, typically in order to compare two fuzzy classifications on an image. An entropy-based criterion has been proposed in [83] for estimating the ability of a feature to characterize and discriminate two classes. However, it does not include spatial information. An alternative would be to generalize the approach developed in [84], where two crisp classifications are compared using a criterion including not only the number of differently classified points but also a spatial distance information.

5. A simple illustrative example

We consider in this section a simple illustrative example, in the fuzzy 2D case. In a magnetic resonance (MR) image of the human brain we have segmented several internal structures using a fuzzy segmentation method. Five fuzzy structures are shown in Fig. 1 (with the standard "left-is-right" convention of medical images): left ventricle (v1), right ventricle (v2), left caudate nucleus (nc1), right caudate nucleus (nc2) and left thalamus (t1).

We have computed distances from all structures to v2, using most of the definitions given in this paper. The

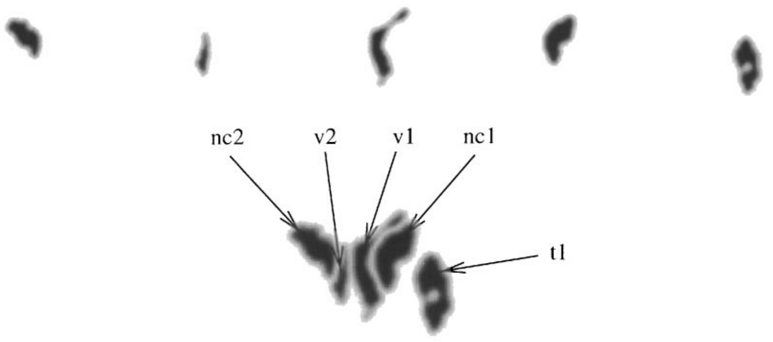


Fig. 1. Top: 5 fuzzy objects resulting from a rough fuzzy segmentation of a MR brain image (membership values rank between 0 and 1, from white to black). Bottom: superposition of these fuzzy objects (the maximum membership value is displayed at each point) and labels as observed in the original MR image.

Table 1
Distances between fuzzy sets using the definitions of Section 3, involving comparison of membership functions only

Distance between v2 and	Distance	nc2	v2	v1	nc1	t1
1	d_∞	1.000	0.000	0.996	1.000	1.000
2	d_1	255.510	0.000	289.718	317.380	322.078
3	d_2	13.348	0.000	13.532	15.054	14.849
4	d_∞ normalized	1.000	0.000	0.996	1.000	1.000
5	d_1 normalized	0.052	0.000	0.059	0.064	0.065
6	d_2 normalized	0.003	0.000	0.003	0.003	0.003
7	Fuzzy entropy	40.831	0.000	127.679	55.575	87.030
8	Normalized fuzzy entropy	0.008	0.000	0.026	0.011	0.018
9	Fuzzy divergence	164.720	0.000	125.834	0.000	0.000
10	Normalized fuzzy divergence	0.033	0.000	0.025	0.000	0.000
11	Pappis (diff/sum)	0.945	0.000	0.957	1.000	1.000
12	1 – sum of t over sum of T (t = min)	0.971	0.000	0.978	1.000	1.000
13	1 – sum of t over sum of T (t = prod)	0.989	0.644	0.992	1.000	1.000
14	1 – sum of t over sum of T (Lukasiewicz)	1.000	0.821	1.000	1.000	1.000
15	1 – Norm. sum of t over T (t = min)	0.126	0.000	0.165	0.163	0.168
16	1 – Norm. sum of t over T (t = prod)	0.129	0.049	0.169	0.163	0.168
17	1 – Norm. sum of t over T (Lukasiewicz)	0.131	0.055	0.170	0.163	0.168
18	1 – Max of inter. (min)	0.667	0.110	0.749	1.000	1.000
19	1 – Max of inter. (product)	0.890	0.208	0.910	1.000	1.000
20	1 – Max of inter. (Lukasiewicz)	1.000	0.220	1.000	1.000	1.000
21	Max of non-inclusion (min)	1.000	0.498	0.996	1.000	1.000
22	Max of non-inclusion (product)	1.000	0.247	0.997	1.000	1.000
23	Max of non-inclusion (Lukasiewicz)	1.000	0.000	0.997	1.000	1.000
24	Norm. sum of non-incl. (min)	0.053	0.011	0.060	0.064	0.065
25	Norm. sum of non-incl. (product)	0.053	0.007	0.060	0.064	0.065
26	Norm. sum of non-incl. (Lukasiewicz)	0.052	0.000	0.059	0.064	0.065

results obtained with the distances of the first class (Section 3) are summarized in Table 1. For definitions involving min and max as intersection and union, we computed the results obtained with extended definitions, using other t-norms and t-conorms. The results using distances of the second class (Section 4) are given in Table 2 for the geometrical approach, and in Table 3 for the morphological approach.

Then we have computed distances from these five structures to a fuzzy model of v2, that has been extracted from another MR image. This model is shown in Fig. 2. It presents differences with v2, but its overall shape is similar to the one of v2. This is typically the kind of examples we have to deal with in model-based pattern recognition. The distances obtained for this model to all structures are shown in Table 4 for the definition of the first class, in Table 5 for the geometrical approach, and in Table 6 for the morphological approach.

Finally, we have computed the distances of 3 points to v2. The coordinates of these points are respectively (25, 40) (point A, with high membership value to v2), (26, 25) (point B, at the border of v2, with low membership value), and (60, 10) (point C, outside of the support of v2). These points are superimposed on v2 in Fig. 3. The results are given in Table 7.

As can be observed from Table 1, the d_p distances (lines 1–6) are not able to differentiate the structures with respect to v2: a value 0 is obtained for v2 (since $d_p(\mu, \mu) = 0$) and for all other objects almost the same value is obtained. The normalization is questionable since it may lead, as in this example, to very low values for all objects. This normalization problem occurs for all other normalized distances in this table.

The problem of the constant value if the supports of μ and ν are disjoint can also be observed: in this example, v2 and nc1 have disjoint supports, as well as v2 and t1. Using these distances, nc1 and t1 have the same distance to v2, although t1 is farther from v2 than nc1 in $\mathcal{S}$.

Since the distance based on fuzzy entropy (lines 7 and 8) does not combine points of μ with points of ν, but only

Fig. 2. A fuzzy model of v2, extracted from another image.

a global measure of the fuzzy sets, made separately, the results can even be counterintuitive. In this example, nc1 has a lower distance to v2 than v1, although v1 is closer to v2 than nc1 in $\mathcal{S}$ (spatially).

For the fuzzy divergence (lines 9 and 10), similar problems occur: some counterintuitive results are obtained, nc1 and t1 have a null distance to v2.

The distances presented in lines 11–23 are not able to differentiate between nc1 and t1, and even v1 and nc2. Very similar values are obtained for all these structures, although they are spatially at very different distances from v2. The property $d(\mu, \mu) = 0$ does not always hold (see lines 13, 14, 16–22).

In lines 24–26, similar problems are observed. Additionally, the normalization leads to very low values for all structures.

Using distances taking into account spatial information, more satisfactory results are obtained. Using the geometrical approach (Table 2), the lowest value is always obtained for v2. A null value is obtained only using the minimum and the Hausdorff distances, since they are the only ones which satisfy $d(\mu, \mu) = 0$. Objects nc2 and v1 have similar distances to v2, as it appears on Fig. 1. Then nc1 is found farther, and then t1. These results fit well the intuition.

The use of different t-norms in the weighted average distance changes the absolute values that are obtained, but not the ranking. Since the following inequalities hold:

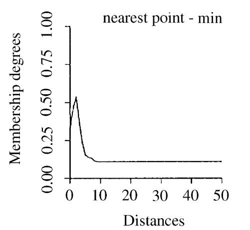
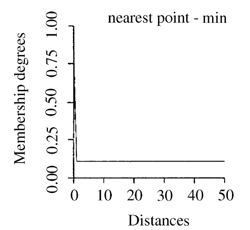
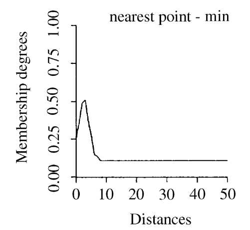
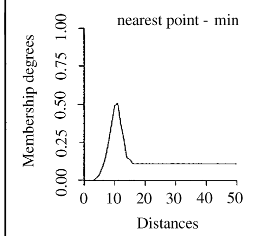
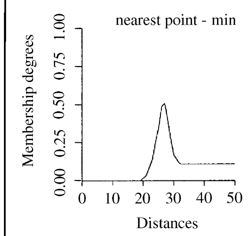
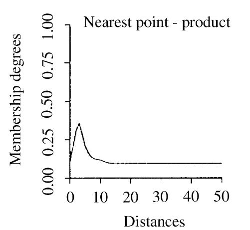
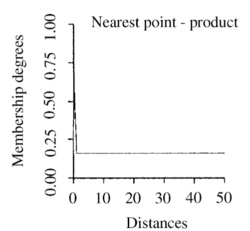
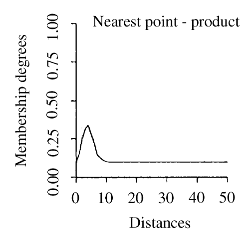
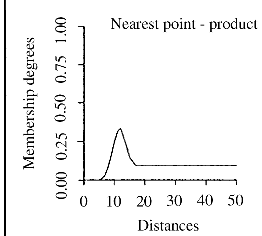
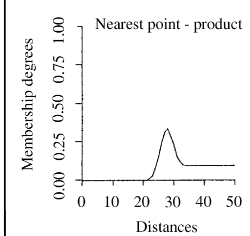
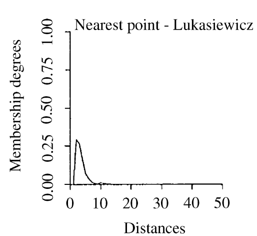
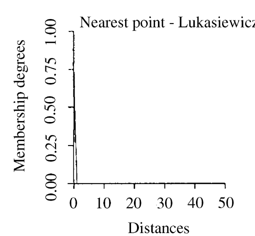
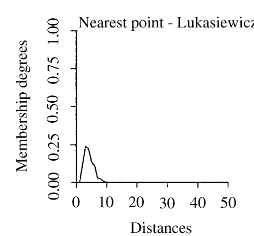
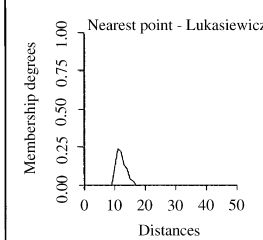
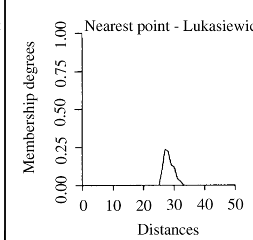
$$\forall (a, b) \in [0, 1]^2, \quad \max(0, a + b - 1) \leqslant ab \leqslant \min(a, b), \tag{54}$$

similar inequalities between the derived distances are obtained. For this distance, the choice of the t-norm is not really important, since it does not change the properties of the distance, and for image processing purposes,

Table 2
Distances between fuzzy sets using the geometrical approach: weighted average distance using 3 different t-norms, fuzzification (using integral over α-cuts) of mean, min and Hausdorff distances

Distance between v2 and Distance	nc2	v2	v1	nc1	t1
Weighted mean dist (min)	16.296	8.165	16.402	22.820	36.762
Weighted mean dist (prod)	16.174	7.501	15.402	22.668	36.589
Weighted mean dist (Lukasiewicz)	16.096	5.855	13.574	22.502	36.299
Integral of mean dist	14.536	5.145	12.897	20.298	32.453
Integral of min dist	1.696	0.000	2.071	8.937	23.204
Integral of Hausdorff dist	19.068	0.000	21.944	25.373	35.952

Table 3
Distances between fuzzy sets using the morphological approach, for the nearest point distance and the Hausdorff distance, using three different t-norms

	Distance between v2 and:				
Distance	nc2	v2	v1	nc1	t1
δ_N (min)					
δ_N (Prod)					
δ_N (Lukasiewicz)					

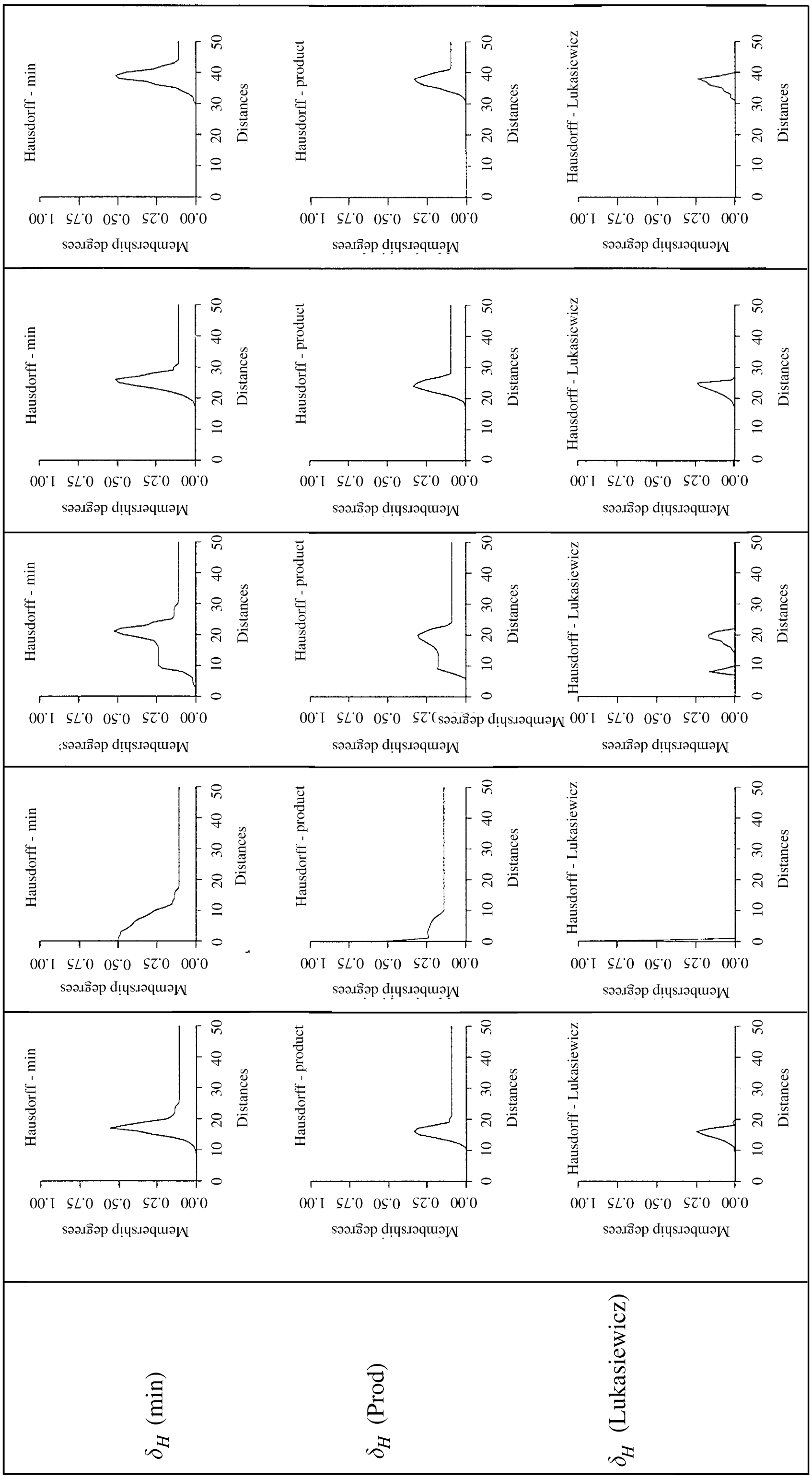

δ_H (min)
δ_H (Prod)
δ_H (Lukasiewicz)
Hausdorff - min
Hausdorff - product
Hausdorff - Lukasiewicz
Membership degrees
Distances

Table 4
Distances between fuzzy sets using the definitions of Section 3, involving comparison of membership functions only. The distance is computed between each of the five structures and a model of v2

Distance between a model of v2 and	Distance	nc2	v2	v1	nc1	t1
1	d_∞	1.000	0.286	0.996	1.000	1.000
2	d_1	258.522	24.494	295.514	328.604	333.302
3	d_2	13.445	1.782	13.686	15.305	15.103
4	d_∞ normalized	1.000	0.286	0.996	1.000	1.000
5	d_1 normalized	0.052	0.005	0.060	0.067	0.067
6	d_2 normalized	0.003	0.000	0.003	0.003	0.003
7	Fuzzy entropy	28.567	12.264	115.415	43.312	74.767
8	Normalized fuzzy entropy	0.006	0.002	0.023	0.009	0.015
9	Fuzzy divergence	231.768	22.065	224.870	0.000	0.000
10	Normalized fuzzy divergence	0.047	0.004	0.046	0.000	0.000
11	Pappis (diff/sum)	0.918	0.147	0.941	1.000	1.000
12	1 – Sum of t over sum of T (t = min)	0.957	0.256	0.970	1.000	1.000
13	1 – Sum of t over sum of T (t = prod)	0.981	0.653	0.990	1.000	1.000
14	1 – Sum of t over sum of T (Lukasiewicz)	1.000	0.815	1.000	1.000	1.000
15	1 – Norm. sum of t over T (t = min)	0.129	0.037	0.167	0.173	0.177
16	1 – Norm. sum of t over T (t = prod)	0.133	0.061	0.172	0.173	0.177
17	1 – Norm. sum of t over T (Lukasiewicz)	0.135	0.066	0.173	0.173	0.177
18	1 – Max of inter. (min)	0.631	0.114	0.631	1.000	1.000
19	1 – Max of inter. (product)	0.863	0.208	0.863	1.000	1.000
20	1 – Max of inter. (Lukasiewicz)	1.000	0.220	1.000	1.000	1.000
21	Max of non-inclusion (min)	1.000	0.616	0.996	1.000	1.000
22	Max of non-inclusion (product)	1.000	0.396	0.996	1.000	1.000
23	Max of non-inclusion (Lukasiewicz)	1.000	0.286	0.996	1.000	1.000
24	Norm. sum of non-incl. (min)	0.055	0.014	0.062	0.067	0.067
25	Norm. sum of non-incl. (product)	0.053	0.011	0.061	0.067	0.067
26	Norm. sum of non-incl. (Lukasiewicz)	0.052	0.005	0.060	0.067	0.067

Table 5
Distances between fuzzy sets using the geometrical approach. The distance is computed between each of the five structures and a model of v2

Distance between a model of v2 and Distance	nc2	v2	v1	nc1	t1
Weighted mean dist (min)	15.851	8.340	16.131	22.354	36.680
Weighted mean dist (prod)	15.627	7.673	15.109	22.138	36.518
Weighted mean dist (Lukasiewicz)	15.381	6.230	13.297	21.870	36.250
Integral of mean dist	14.234	5.705	12.853	20.232	33.498
Integral of min dist	1.664	0.004	1.875	8.820	23.638
Integral of Hausdorff dist	18.099	2.683	21.777	25.437	36.642

Table 6
Distances between fuzzy sets using the morphological approach, for the nearest point distance and the Hausdorff distance, using 3 different t-norms. The distance is computed between each of the five structures and a model of v2

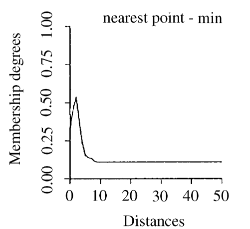
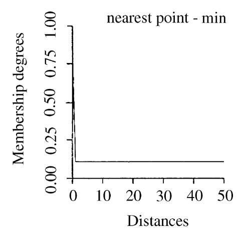
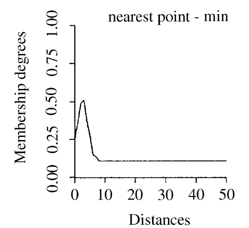
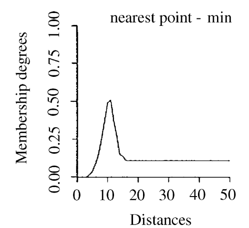
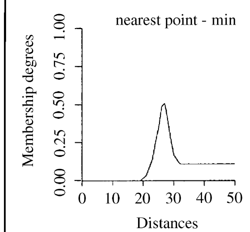
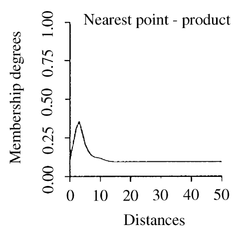
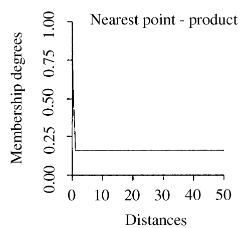
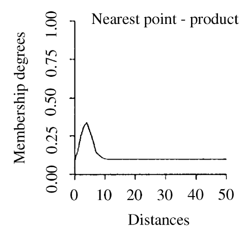
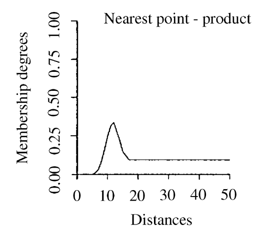
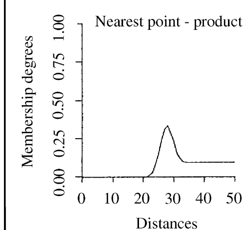
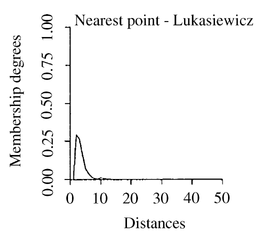
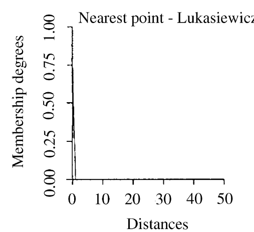
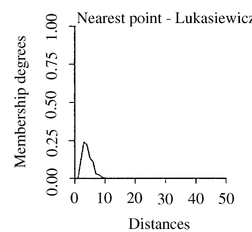
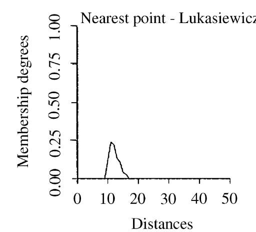
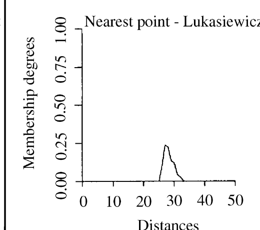
	Distance between v2 and:				
Distance	nc2	v2	v1	nc1	t1
δ_N (min)					
δ_N (Prod)					
δ_N (Lukasiewicz)					

Table 6 (*Continued*)

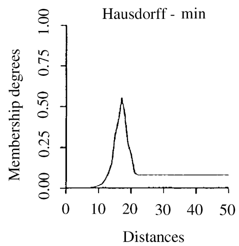
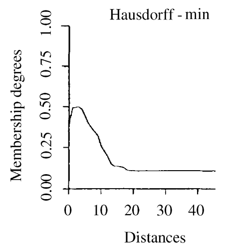
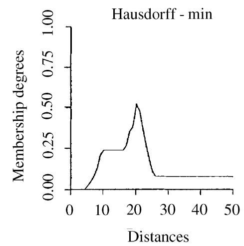
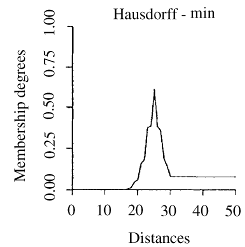
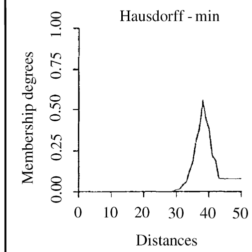
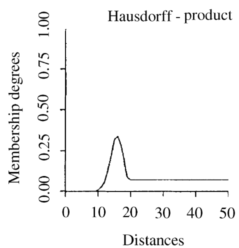
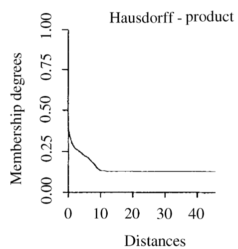
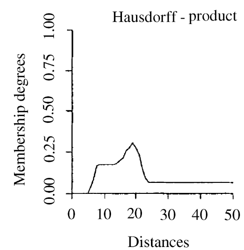
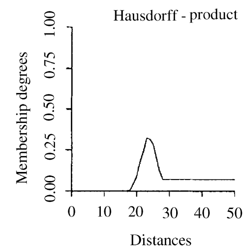
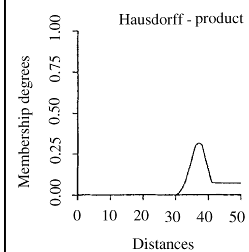
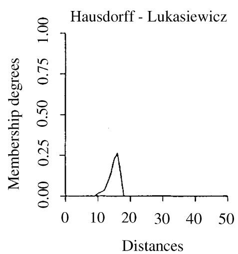
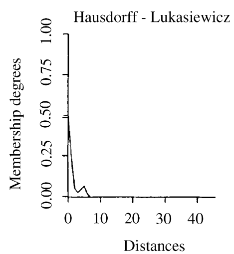
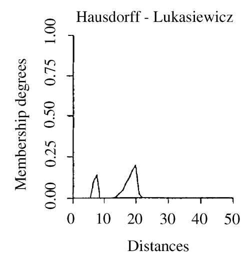
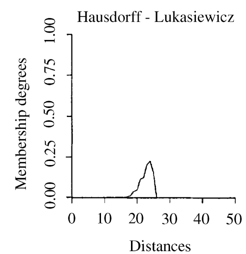
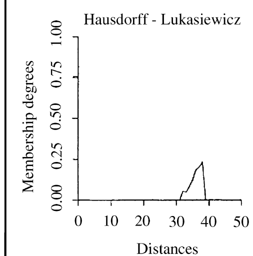
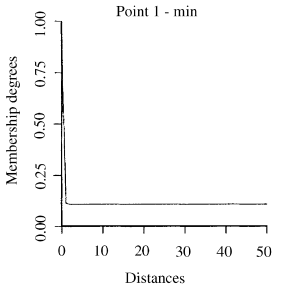
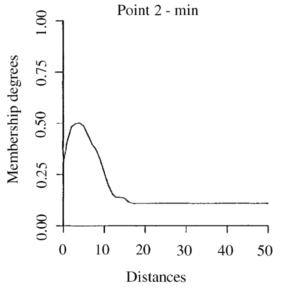
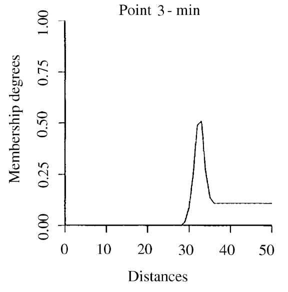
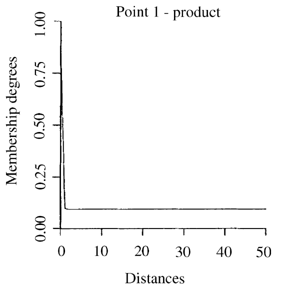
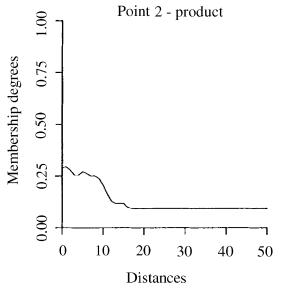
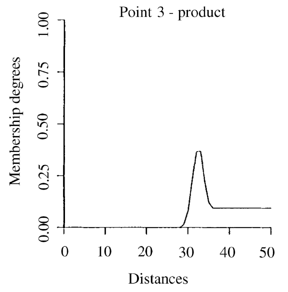
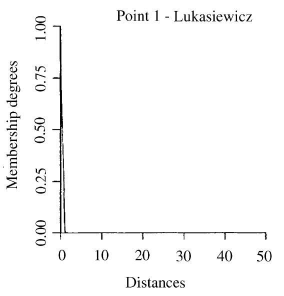
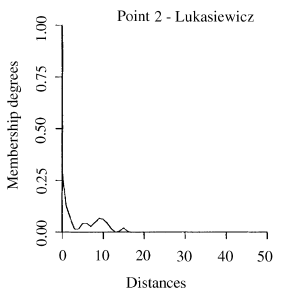
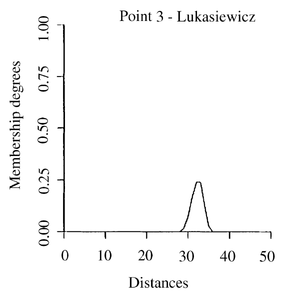
Distance	Distance between v2 and: nc2	v2	v1	nc1	t1
δ_H (min)					
δ_H (Prod)					
δ_H (Luckasiewicz)					

Table 7
Distance from a point to a fuzzy set: example of three points and v2, with three different t-norms

Distance	Point (25, 40) to v2	Point (26, 25) to v2	Point (60, 10) to v2
t-norm: min			
t-norm: product			
t-norm: Lukasiewicz			

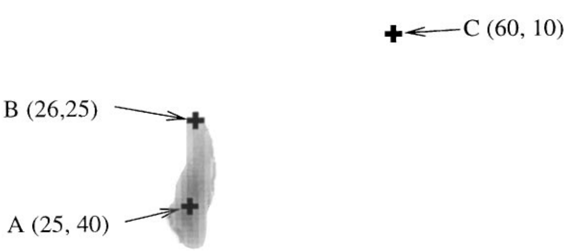


Fig. 3. The three points used in the presented example with respect to v2.

the ranking between distance values is often more important than their absolute value.

All previous examples provide results as numbers. When using the morphological approach, the results take the form of fuzzy numbers as seen in Table 3. The curves in this table show the degrees to which the distance is equal to n as a function of n. Again the results fit well the intuition. The distributions obtained for v2 are concentrated on the low distance values. Then, when the structures become farther from v2, the curves are shifted towards higher distance values.

Here again the choice of a specific t-norm is not crucial as it changes mainly the absolute values. Lower membership degrees are obtained when using a smaller t-norm.

The fact that the Hausdorff distance provides higher values than the nearest point distance corresponds to the fact that the size of the dilation applied to one set needed to reach the other is less than the size of the dilation needed to completely include the other set. This is the case for crisp sets, and the same property holds in the fuzzy case.

Tables 4–6 correspond to the case where, starting from a model object, we are looking among a set of objects if one of them corresponds to the model. This contrasts with Tables 1–3, where we try to assess the distance between objects of the same image. We cannot expect to have a distance between v2 and the model strictly equal to 0, however, we expect lower values than between the model and any of the other structures. This is indeed observed for several distances of Tables 4. The d_p distances provide satisfactory results in this respect (lines 1–3). The normalized d_p distances (lines 4–6) are not as satisfactory, because of the normalization, leading to differences between values that are too low to be really discriminating.

Entropy and divergence approaches (lines 7–10) are not satisfactory and lead to counter-intuitive results as before.

The distances in lines 11–23 provide satisfactory results. Lower values are obtained for v2 than for the other structures. The fact that these other structures are not discriminated is less important for this type of problem. It is enough to know that none of them matches the model.

The last distances (lines 24–26) suffer from the normalization problem. Although lower values are obtained for v2, they are not very different from the others.

The geometrical distances presented in Table 5 provide good results, very similar to those obtained in Table 2. We do not obtain a null value for v2 using the nearest point and the Hausdorff distances, since v2 does not perfectly match the model, but we obtain values that are still much lower than those obtained for the other structures.

The morphological approach in Table 6 also provides good results, similar to those obtained in Table 3. Small variations can be observed in the curves, but they are not really significant. The curves are a little bit more spread for v2 than in Table 3, as expected since the match is not perfect.

Finally, considering the distance of a point to a fuzzy set (to v2 in Table 7), defined as a fuzzy number, we again observe satisfactory results. For the first point, which has a high membership to the fuzzy set, the distributions take a high value at 0 (equal to $\mu(x)$), and decrease very fast.

For the second point, which belongs to μ with a low membership value, the distributions are more spread. This represents the ambiguity in defining the distance of this point to the fuzzy set. For instance if we consider some defuzzification process using a threshold value of μ, depending on this threshold, the point would be more or less close to μ.

The third point is outside of the support of μ, therefore the membership degrees of low distances are all equal to 0, and the distributions are shifted towards higher values.

6. Discussion and conclusion

Among the three kinds of distances that can be considered in fuzzy sets, we mainly discussed in this paper the widely used distance between two fuzzy sets, for which several definitions exist. The large variety of such distances has been presented from an image processing point of view, following the type of information they convey. We proposed also some generalizations and a new morphological approach. We underlined some advantages of all these distances for image processing and pattern recognition. Their use needs now to be further investigated in this domain.

Two classes of distances have been defined, those comparing mainly the membership functions, and those accounting also for spatial distances. Most definitions we found in the literature belong to the first class. This is mainly due to the fact that they have been developed for other applications than image processing.

In the first class of methods, the only way μ and ν are combined is by computation linking $\mu(x)$ and $\nu(x)$, i.e. only the memberships at the same point of $\mathscr{S}$. No spatial information is taken into account. A positive consequence is that the corresponding distances are easy to compute. The complexity is linear in the cardinality of $\mathscr{S}$. Considering image processing applications, we suggest that the first class of methods (comparing membership functions only) be restricted when the two fuzzy sets to be compared represent the same structure or a structure and a model. Applications in model-based or case-based pattern recognition are foreseeable.

On the other hand, the definitions which combine spatial distances and fuzzy membership comparison allow for a more general analysis of structures in images, for applications where topological and spatial arrangement of the structures of interest is important (segmentation, classification, scene interpretation). This is permitted by the fact that these distances combine membership values at different points in the space, therefore taking into account their proximity or farness in $\mathscr{S}$. The price to pay is an increased complexity, generally quadratic in the cardinality of $\mathscr{S}$.

When facing the problem of choosing a distance, several criteria can be used. First, the type of application at hand plays an important role. While both classes of methods can be used for comparing an object and a model object, only the second class can be used for evaluating distances between objects in the same image. Among the distances of the first class, the results we obtained show that entropy and divergence based approaches are not satisfactory. Also normalized distances should be avoided in most cases. The choice among the remaining distances can be done by looking at the properties of the distances (for instance, do we need $d(\mu, \mu) = 0$ for the application at hand?), and at the computation time. Among the distances of the second class, similar choice criteria can be used.

It is noticeable that most of these definitions are found in other domains than image processing. We did not find much applications in image processing. Among these few applications, most of them deal with fuzzy sets representing features extracted from the images, and not directly spatial image objects. This may be explained by the fact that less definitions accounting for spatial information are available. However, we argue that this is an interesting field for future research.

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