Fuzzy Relative Position Between Objects in Image Processing: A Morphological Approach

Isabelle Bloch, *Member*, *IEEE*

Abstract—In order to cope with the ambiguity of spatial relative position concepts, we propose a new definition of the relative position between two objects in a fuzzy set framework. This definition is based on a morphological and fuzzy pattern-matching approach, and consists of comparing an object to a fuzzy landscape representing the degree of satisfaction of a directional relationship to a reference object. It has good formal properties, it is flexible, it fits the intuition, and it can be used for structural pattern recognition under imprecision. Moreover, it also applies in 3D and for fuzzy objects issued from images.

Index Terms—Fuzzy sets, spatial relative position, directional relations, fuzzy mathematical morphology, structural shape recognition.

1 Introduction

The spatial arrangement of objects in images provides important information for recognition and interpretation tasks, in particular when the objects are embedded in a complex environment like in medical or remote sensing images. Relationships between objects can be partly described in terms of relative position, like "to the left of," and it is the aim of this paper to address the problem of defining such relationships. It should be noted that such concepts are rather ambiguous, they defy precise definitions, but human beings have a rather intuitive and common way of understanding and interpreting them. From our every day experience, it is clear that any "all-or-nothing" definition leads to unsatisfactory results in several situations, even of moderate complexity (see examples of Fig. 1).

Therefore, relative position concepts may find a better understanding in the framework of fuzzy sets, as fuzzy relationships. This framework makes it possible to propose flexible definitions which fit the intuition and may include subjective aspects, depending on the application and on the requirements of the user. The interest of fuzzy approaches for representing spatial constraints has been emphasized e.g., in [9]. Fuzzy approaches are all the most interesting when imprecision in images has to be taken into account. Indeed, the representation of image regions as spatial fuzzy sets is useful to take into account the imprecision inherent to images. The applications that are anticipated from this work are related to structural pattern recognition, where we are not just interested in the dominating relationships between objects: an object may satisfy several relationships with respect to the other components of the image (see e.g., Fig. 1, right) and it is clear that the shape of the considered objects has to play an important role in assessing its relative position.

The problem of defining relative positioning has already been addressed in the literature. To our knowledge, almost all existing methods for defining fuzzy relative spatial position rely on angle measurements between points of the two objects of interest [13], [15], [10], and concern 2D objects. In these approaches, a fuzzy relationship is defined as a fuzzy set. More precisely, a relative position relationship is defined as a linguistic variable and is represented as a fuzzy set depending on an angle θ. On the objects, the angle $\theta(a, b)$ is measured between the segment joining two points a and b and the x-axis of the coordinate frame. Then the agreement between the relation and the measured angles is evaluated, according to three possible methods:

1) representing each object by a characteristic point as in [13], [10], or
2) using an aggregation method in [13], [10], or
3) using a compatibility method [15], which consists in defining a fuzzy set in [0, 1] representing the compatibility between the normalized angle histogram and the fuzzy relation.

Another method, based on a different principle, has been proposed recently in [14]. It relies on a histogram of forces (and not of angles) computed from the intersections of the objects with lines having the desired direction. Finally, the method described in [11] consists in defining a fuzzy area, "left from A" for instance, from a projection of the object A on the horizontal axis. The degree to which B is to the left from A results from a combination of the degree of projection of B and the membership degree of B in the fuzzy area.

In order to include real information on object shapes, we propose in this paper an original approach, completely different from the previous ones. The main idea is to base the computation on a morphological approach, together with a fuzzy pattern-matching procedure, directly in the image space, and providing the relative position between two objects in any direction. We suggested this idea in [1]. In this paper, we state this method by proposing original definitions in Section 2, and we provide improvements over what is described in [1] as well as an in depth study of the properties and behavior of obtained relationships (Section 3). Examples of real images are presented in Section 4. In particular, the proposed definition is examined under the light of its generality, since it applies to 3D objects, to fuzzy objects, and leads to the assessment of relative position in any direction.

2 A New Definition of Relative Position Between Two Objects

2.1 Overview: Morphological Fuzzy Pattern-Matching Approach

Our motivation for proposing a new definition for relative position between objects is to provide a definition that should: avoid angle histogram computation in order to reduce the computational burden; be generic enough in order to apply to relative positions defined by any direction (not only four basic ones); introduce morphological information on the considered objects themselves (not only intersecting segments or projections), with the aim of pattern-recognition applications; be applicable to 3D objects (for applications to medical imaging for instance) and to fuzzy objects (in order to take into account spatial imprecision in the objects); verify algebraic and geometrical properties and behave according to the intuition in a large variety of situations.

Let us consider a reference object R and an object A for which the relative position with respect to R has to be evaluated. In order to evaluate the degree to which A is in some direction with respect to R, we propose the following approach:

1) We first define a fuzzy "landscape" around the reference object R as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the

• *The author is with Ecole Nationale Supérieure des Télécommunications, Département TSI–CNRS URA 820, 46 rue Barrault, 75634 Paris, Cedex 13, France. E-mail: bloch@ima.enst.fr.*

Manuscript received 30 Apr. 1997; revised 1 Feb. 1999. Recommended for acceptance by I. Sethi.

For information on obtaining reprints of this article, please send e-mail to: tpami@computer.org, and reference IEEECS Log Number 107715.

0162-8828/99/$10.00 © 1999 IEEE

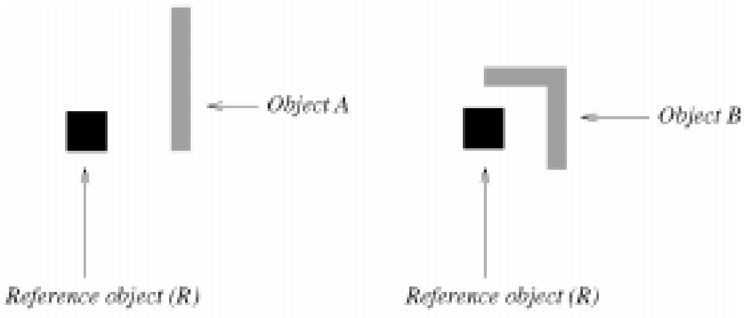


Fig. 1. Two examples where the relative position of objects with respect to the reference object is difficult to define in an "all-or-nothing" manner: Object A is to the right of R, but it can also be considered to be to some extent above it; Object B is strongly to the right of R and above it.

spatial relation under examination. We make use here of a spatial representation of fuzzy sets, which already proved to be useful in image processing [12], [3]. Therefore, the fuzzy landscape is directly defined in the same space as the considered objects, in the contrary to the solution proposed in [11], where the fuzzy area is defined on a one-dimensional axis, by using projections of the objects.

2) We then compare the object A to the fuzzy landscape attached to R, in order to evaluate how well the object matches with the areas having high membership values (i.e., areas that are in the desired direction). This is done using a fuzzy pattern-matching approach, which provides an evaluation as an interval instead of one number only. This makes a major difference with respect to all the previous approaches, and, to our opinion, it provides a richer information about the considered relationship.

2.2 Relative Position From Fuzzy Pattern Matching

We denote by S the Euclidean space where the objects are defined. S is typically a 2D or 3D discrete space (as in image processing). In the 3D Euclidean space, a direction is defined by two angles α_1 and α_2, where

$$\alpha_1 \in [0, 2\pi] \text{ and } \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

($\alpha_2 = 0$, in the 2D case). The direction in which the relative position of an object with respect to another one is evaluated is denoted by:

$$\vec{u}_{\alpha_1,\alpha_2} = (\cos\alpha_2 \cos\alpha_1, \cos\alpha_2 \sin\alpha_1, \sin\alpha_2)^t,$$

and we note $\alpha = (\alpha_1, \alpha_2)$.

We consider two (possibly fuzzy) objects, R and A, and define the degree to which A is in direction

$$\vec{u}_{\alpha_1,\alpha_2}$$

with respect to R. Let us denote by $\mu_\alpha(R)$ the fuzzy set defined in the image in such a way that points of areas which satisfy to a high degree the relation "to be in the direction

$$\vec{u}_{\alpha_1,\alpha_2}$$

with respect to reference object R" have high membership values. In other terms, the membership function $\mu_\alpha(R)$ has to be an increasing function of the degree of satisfaction of the relation. It is a spatial fuzzy set (i.e., a function of the image S into [0, 1]) and directly related to the shape of R. The precise definition of $\mu_\alpha(R)$ is given in the next subsection.

Let us denote by μ_A the membership function of the object A, which is a function of S into [0, 1]. The evaluation of relative position of A with respect to R is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all x in S. An appropriate tool for defining this function is the fuzzy pattern-matching approach [8]. Following this approach, the evaluation of the matching between two possibility distributions consists of two numbers, a necessity degree N (a pessimistic evaluation) and a possibility degree Π (an optimistic evaluation), as often used in the fuzzy set community. In our application, they take the following forms:

$$\prod_{\alpha_1,\alpha_2}^{R}(A) = \sup_{x \in S} t\left[\mu_\alpha(R)(x),\ \mu_A(x)\right],$$
$$N_{\alpha_1,\alpha_2}^{R}(A) = \inf_{x \in S} T\left[\mu_\alpha(R)(x),\ 1-\mu_A(x)\right] \tag{1}$$

where t is a t-norm (fuzzy intersection) and T a t-conorm (fuzzy union) [7]. In the crisp case, these equations reduce to:

$$\prod_{\alpha_1,\alpha_2}^{R}(A) = \sup_{x \in A} \mu_\alpha(R)(x),$$

and

$$N_{\alpha_1,\alpha_2}^{R}(A) = \inf_{x \in A} \mu_\alpha(R)(x).$$

The possibility corresponds to a degree of intersection between the fuzzy sets A and $\mu_\alpha(R)$, while the necessity corresponds to a degree of inclusion of A in $\mu_\alpha(R)$. They can also be interpreted in terms of fuzzy mathematical morphology, since the possibility

$$\prod_{\alpha_1,\alpha_2}^{R}(A)$$

is equal to the dilation of μ_A by $\mu_\alpha(R)$ at the origin, while the necessity

$$N_{\alpha_1,\alpha_2}^{R}(A)$$

is equal to the erosion, as shown in [5]. These two interpretations, in terms of set theoretic operations and in terms of morphological ones, explain how the shape of the objects is taken into account.

Several other functions combining $\mu_\alpha(R)$ and $\mu_A(x)$ can be constructed. The extreme values provided by the fuzzy pattern matching are interesting because of their morphological interpretation, and because they provide an interval and not only a single value and may represent in this way the ambiguity of the relation if any. An average measure can also be useful from a practical point of view, and is defined as:

$$M_{\alpha_1,\alpha_2}^{R}(A) = \frac{1}{|A|}\sum_{x \in S} \mu_A(x)\mu_\alpha(R)(x), \tag{2}$$

where $|A|$ denotes the fuzzy cardinality of A:

$$|A| = \sum_{x \in S} \mu_A(x).$$

2.3 Definition of μ_α

The key point in the previous definition relies in the definition of $\mu_\alpha(R)$. The requirements stated above for this fuzzy set are not strong and leave room for a large spectrum of possibilities. This flexibility allows the user to define any membership function according to the application at hand and the context requirements. We propose here a definition that looks precisely at the domains of space that are visible from a reference object point in the direction

$$\vec{u}_{\alpha_1,\alpha_2}.$$

This applies to objects of any kind, in particular having strong concavities and, therefore, differs from solutions proposed in [1], [2].

Let us denote by P any point in S, and by Q any point in R. Let $\beta(P, Q)$ be the angle between the vector

$$\overrightarrow{QP}$$

and the direction

$$\vec{u}_{\alpha_1,\alpha_2},$$

computed in $[0, \pi]$:

$$\beta(P,Q) = \arccos\left[\frac{\overrightarrow{QP} \cdot \vec{u}_{\alpha_1,\alpha_2}}{\left\|\overrightarrow{QP}\right\|}\right], \text{ and } \beta(P,P) = 0. \tag{3}$$

We then determine for each point P the point Q of R leading to the smallest angle β, denoted by $\beta_{\min}$. In the crisp case, this point Q is the reference object point from which P is visible in the direction the closest to

$$\vec{u}_{\alpha_1,\alpha_2}$$

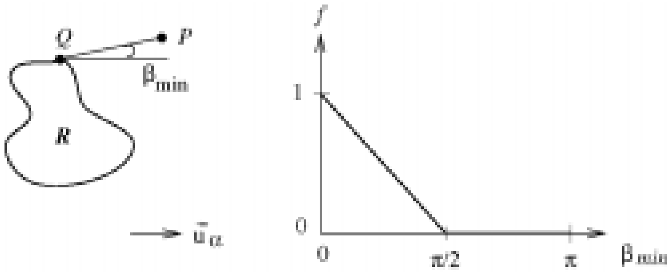
(see Fig. 2): $\beta_{\min}(P) = \min_{Q \in R} \beta(P, Q)$. The fuzzy landscape $\mu_\alpha(R)$ at point P is then defined as: $\mu_\alpha(R)(P) = f(\beta_{\min}(P))$, where f is a decreasing function of $[0, \pi]$ into $[0, 1]$. In our experiments, we have chosen a simple linear function (Fig. 2b):

$$\mu_\alpha(R)(P) = \max\left(0,\ 1 - \frac{2\beta_{\min}(P)}{\pi}\right).$$

Illustrations of the definition of $\mu_\alpha(R)$ are given in Fig. 3 for several reference objects. They show the consistency of the proposed approach in case of concavities: since the aim of the proposed definition is not to find only the dominant relationship, an object may satisfy several different relationships with high degrees. Therefore, "to be to the right of R" does not mean that the object should be completely to the right of the reference object, but only that it is at least to the right of some part of it. This is the case, for instance, in Fig. 3, where we obtain high values of being right inside the concavities (note that we obtain high values for any direction inside the hole, which is actually a way to model the fact that an object is in the hole using directional relationships).

In the fuzzy case, we propose a method which avoids combining directly an angle with a membership value, and only combines membership values, one describing the membership to R, and the other to the fuzzy landscape. This corresponds to translating binary equations and propositions into fuzzy ones: in the binary case, we express that: $Q \in R$ and $f(\beta_{\min}) = \max_{Q \in R} f(\beta(P, Q))$ (since f is decreasing), which translates in fuzzy terms as:

$$\mu_\alpha(R)(P) = \max_{Q \in Supp(S)} t\left[\mu_R(Q), f\left(\beta(P,Q)\right)\right], \tag{4}$$



(a) (b)

Fig. 2. Definition of (a) $\beta_{\min}$ and (b) $f(\beta_{\min})$.

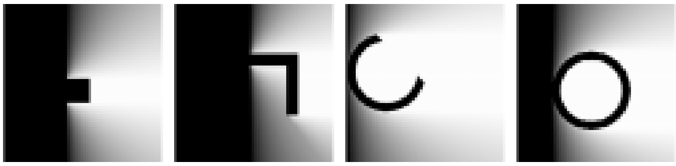
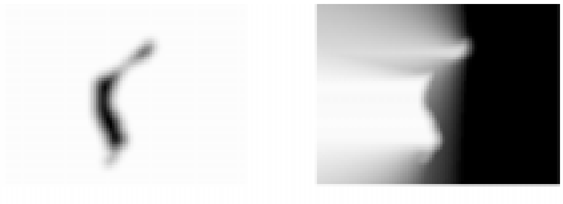


Fig. 3. A few examples of $\mu_\alpha(R)$ for $\alpha_1 = \alpha_2 = 0$ corresponding to the relative position "right" (high gray values correspond to high membership values) using the angle of visibility method, for different types of reference objects (reference objects are black).



(a) (b)

Fig. 4. (a) A fuzzy reference object. (b) Fuzzy landscape representing the relationship "to the left of" for the proposed combination method (4).

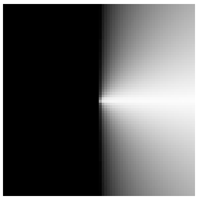


Fig. 5. Structuring element ν for $\alpha_1 = 0$ (high gray values correspond to high membership values).

where t is a t-norm. Fig. 4 illustrates the obtained result on a fuzzy object.

An advantage of this approach is its interpretation in terms of morphological operations. It can be shown that $\mu_\alpha(R)$ is exactly the fuzzy dilation of μ_R by ν, where ν is a fuzzy structuring element defined on S as:

$$\forall P \in S,\ \nu(P) = \max\left[0,\ 1 - \frac{2}{\pi}\arccos\frac{\overrightarrow{OP} \cdot \vec{u}_\alpha}{\left\|\overrightarrow{OP}\right\|}\right], \tag{5}$$

where O is the center of the structuring element. This structuring element is illustrated in 2D in Fig. 5. The following definition is used for the fuzzy dilation (see [5] for more details about fuzzy morphological operations):

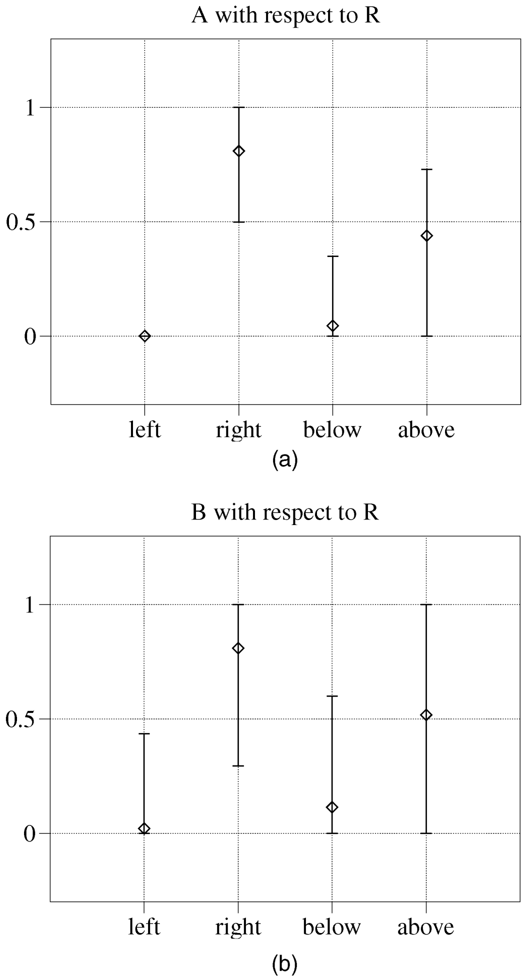


Fig. 6. (a) Results obtained for the object *A* of Fig. 1 with respect to reference object *R*. The three given values correspond to necessity (lowest value of the bar) and possibility (highest value of the bar) degrees, and to the average value (diamond). (b) Results obtained for the object *B* of Fig. 1 with respect to reference object *R*.

$$\forall P \in S,\ D_\nu(\mu)(P) = \max_{Q \in S} t\left[\mu(Q),\ \nu(P-Q)\right] \tag{6}$$

where t is a t-norm. This equivalence provides an additional morphological interpretation of our definition.

2.4 Two Simple Examples

We illustrate the proposed definition on the two simple 2D examples shown in Fig. 1, and compute the relative position of objects A (rectangle) and B (corner) with respect to reference object R (square), for four directions:

- left ($\alpha_1 = \pi$),
- right ($\alpha_1 = 0$),
- above ($\alpha_1 = \pi/2$), and
- below ($\alpha_1 = 3\pi/2$).

Fig. 6 provides the obtained results, using (1) and (2) for the possibility degree, necessity degree and average, respectively. The interval $[N, \Pi]$ represents the range between the minimal and maximal values obtained in the object for the degrees of satisfaction of the relation to the reference object. This can be interpreted as the ambiguity of the relationship, or as the ignorance we have about a precise value we could give for this relation.

These results fit well the intuition. Object A is found mainly on the right of reference object R and to some extent also above it. The lower part of A is not above R and, therefore, the necessity for this relation is equal to 0. Similarly, B is found to be mainly on the right and above object R. This last relation is even more ambiguous than in the case of A, since a part of B is completely above R while another is completely not above it. We obtain in this case the maximum ambiguity, represented by an interval of 1 between necessity and possibility.

The average values provide a summary of the satisfaction of the relationship. Of course it is but one possible way to provide a global measure. Other measures could be derived from the set of all values taken by $\mu_\alpha(R)(P)$ for P belonging to the considered object.

2.5 Implementation Issues

The direct computation of the proposed formula for $\beta_{\min}$, with an exhaustive method (search over all points Q), can be computationally very expensive in the 3D case. The computation can be made faster by storing the list of points in R (which are often much less numerous than all image points), and by tabulating angles (since

$$\overrightarrow{QP}$$

takes a finite number of integer values in discrete images). The interpretation of the proposed definition as a fuzzy dilation may suggest a further way to reduce the computation time by reducing the precision of $\mu_\alpha(R)$: It consists in performing the fuzzy dilation with a limited support for the structuring element. This amounts to have a rough quantification of angles and, therefore, an approximate result is obtained.

2.5.1 Propagation Algorithm

We propose here a fast algorithm for computing $\mu_\alpha(R)$, that still provides an approximation of $\mu_\alpha(R)$ but with increased precision with respect to the algorithm based on dilation. This algorithm is based on a propagation technique inspired by chamfer methods used for instance for discrete distance computation [6]. This idea comes from the observation of Fig. 3 where it appears that membership values in the fuzzy set μ_α are constant along lines issued from contour points of the reference object.

The algorithm consists in performing two passes on the image, one in the conventional sense, and one in the opposite sense. For each point P, we store the point $Q = O(P)$ from which the minimum visibility angle is obtained. For a point P, we do not consider all points in R as for the exhaustive method, but only those of a neighborhood of P. The algorithm consists of the following steps:

1) Initialization: We set $O(P) = P$ if $P \in R$ and $O(P) = Null$ otherwise.
2) First pass: We compute the fuzzy landscape from visibility angle at P as:
   - $\mu_\alpha(R)(P) = \max_{Q \in V(P)} t[\mu_R(O(Q)),\ f(\beta(P,\ O(Q)))]$, where $V(P)$ denotes the neighborhood of P. Let Q_P be the point Q for which the maximum value is obtained:

   $$Q_P = argmax_{Q \in V(P)} t[\mu_R(O(Q)), f(\beta(P, O(Q)))].$$

   Then, we set: $O(P) = O(Q_P)$.
3) Second pass: It is performed as the first one, except that the points are examined in the reverse order. Note that during these two passes, the points of R can also be modified.

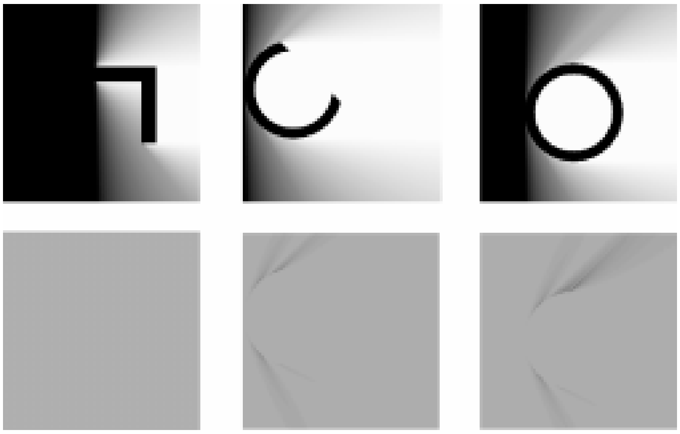


Fig. 7. A few examples of $\mu_\alpha(R)$ for $\alpha_1 = \alpha_2 = 0$ for different types of reference objects (reference objects are black) using the propagation method. The bottom row shows the difference with the exact method (a gray level of 128 corresponds to no error, and the differences have been enhanced for the visualization). For the corner (left example), we obtain no error for all directions.

This algorithm is applicable in 2D as well as in 3D, and for crisp objects as well as for fuzzy ones. We used 8-connectivity in 2D, and 26-connectivity in 3D for defining $V(P)$. More precise results could be obtained with larger neighborhoods or with more passes on the image using other propagation directions, but at the price of extended computation time. The errors are mainly due to the fact that when there are several candidates for Q_P (i.e., leading to the same minimal value for $\beta_{\min}$), there is no clear strategy of choice of one particular point among the candidates.

2.5.2 Results

Although the result obtained for $\mu_\alpha(R)$ using the propagation algorithm is not exact, it can be considered as a good approximation. Fig. 7 illustrates the results obtained with the propagation algorithm and the difference with the exact method for several reference objects. They show the quality of the approximation. The results may show no error at all depending on the angle with respect to the propagation directions, and depending on the object (this is the case, for instance, for the square of Fig. 1). In the fuzzy case too, only few differences can be observed. Moreover, when using these results instead of the exact ones, we observed only few differences in the pattern-matching results (the maximum error is at most a few percentage points, and generally less than 5 percent). These differences cannot be considered as of much significance for pattern-recognition purposes.

2.5.3 Complexity Analysis

Let us denote by N the number of points in the image, n_R the number of points in the reference object R, and n_A the number of points in the object A (in their support in the fuzzy case), and n_V the number of points in the considered neighborhood. The complexity of the exhaustive method for computing $\mu_\alpha(R)$ is: $O(n_R N)$. If we limit the computation to points of A, then it becomes only $O(n_R n_A)$. With the propagation method, the complexity is as follows: $O(N)$ for the initialization (step 1), $O(n_V)$ for each point for step 2, and $O(n_V)$ for each point for step 3. Finally, we obtain for the complete algorithm: $O((1 + 2n_V)N)$. Step 1 consists just in setting values; therefore, the computation for each point is constant and very low. In steps 2 and 3, the computation for each of the n_V points includes computation of $f(\beta)$, which is constant and of low cost since angles are tabulated, and computation of a t-norm, which is also constant and of low cost. By using the propagation algorithm, the factor gained over the exhaustive method is:

$$\frac{n_R}{1 + 2n_V}.$$

In our experiments, where we have $n_V = 8$ or $n_V = 26$, we observed that the propagation algorithm may run more than 20 times faster than the exhaustive method. If we limit the computation of $\mu_\alpha(R)$ to the points of A with the exhaustive method, the gain is:

$$\frac{n_R n_A}{\left(1 + 2n_V\right)N},$$

which may be less than 1, depending on the objects. Finally, we observe that the propagation algorithm is very suitable in particular when the position of several objects has to be assessed with respect to a reference object R. In such cases, we can take advantage of a preliminary computation of $\mu_\alpha(R)$ in the whole image since $\mu_\alpha(R)$ depends only on R. In this way $\mu_\alpha(R)$ has to be computed only once. Note that the additionnal complexity to compute the necessity and possibility measures is only $O(n_A)$.

3 Properties of the Morphological Definition

In this section, we investigate some properties of our approach. We first look at some theoretical results concerning the algebraic behavior of the proposed definition. Then we describe some geometrical properties and behavior.

3.1 Algebraic Properties

We proved two algebraic properties of fuzzy relative position, reflexivity and symmetry. The symmetry property, corresponding to the fact that if A is left to B, we expect B to be right to A, is certainly desirable for pattern-recognition applications. On the contrary, it is not as clear if reflexivity is a real requirement for such applications. Transitivity will be addressed shortly in the conclusion, as part of an inference tool for spatial reasoning.

When defining

$$\mu_{\alpha_1,\alpha_2}(R),$$

we have two choices for the points of R. The first one is to take

$$\mu_{\alpha_1,\alpha_2}(R)(P) = 1$$

for those points, the second one being to define $\mu_\alpha(R)$ only on $S - R$. In the first case, the degree of satisfaction of a relationship is reflexive, in the sense that any crisp object totally satisfies any relation with itself:

$$\forall \alpha_1 \in [0, 2\pi],\ \forall \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$\prod_{\alpha_1,\alpha_2}^{R}(R) = N^R_{\alpha_1,\alpha_2}(R) = M^R_{\alpha_1,\alpha_2}(R) = 1 \tag{7}$$

In the fuzzy case, we have:

$$\forall \alpha_1 \in [0, 2\pi],\ \forall \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$\prod_{\alpha_1,\alpha_2}^{R}(R) = \max_{Q \in Supp(R)} \mu_R(Q), \tag{8}$$

which is equal to 1 if R is a normalized fuzzy set. Thus, the reflexivity property holds for the possibility; nothing can be said in general for necessity and average.

For the possibility, the following symmetry property holds (in the crisp case as well as in the fuzzy case):

$$\forall \alpha_1 \in [0, 2\pi], \forall \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$\prod_{\alpha_1,\alpha_2}^{R}(A) = \prod_{\pi+\alpha_1,-\alpha_2}^{A}(R). \quad (9)$$

Proofs can be found in [4]. In the 2D case, this reduces to:

$$\prod_{\alpha}^{R}(A) = \prod_{\pi+\alpha}^{A}(R).$$

3.2 Geometrical Properties

The proposed definition is invariant with respect to translation, rotation and scaling, for 2D and 3D objects (crisp and fuzzy). If we denote by τ any translation, r any rotation, and λ any scaling factor (in $\mathbb{R}^+$), we proved that [4]:

$$\forall \alpha_1 \in [0, 2\pi], \forall \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\prod_{\alpha_1,\alpha_2}^{\tau(R)}[\tau(A)] = \prod_{\alpha_1,\alpha_2}^{R}(A),$$
$$\prod_{r(\alpha_1,\alpha_2)}^{r(R)}[r(A)] = \prod_{\alpha_1,\alpha_2}^{R}(A),$$
$$\prod_{\alpha_1,\alpha_2}^{\lambda R}(\lambda A) = \prod_{\alpha_1,\alpha_2}^{R}(A), \quad (10)$$

$$N_{\alpha_1,\alpha_2}^{\tau(R)}[\tau(A)] = N_{\alpha_1,\alpha_2}^{R}(A),$$
$$N_{r(\alpha_1,\alpha_2)}^{r(R)}[r(A)] = N_{\alpha_1,\alpha_2}^{R}(A),$$
$$N_{\alpha_1,\alpha_2}^{\lambda R}(\lambda A) = N_{\alpha_1,\alpha_2}^{R}(A), \quad (11)$$

$$M_{\alpha_1,\alpha_2}^{\tau(R)}[\tau(A)] = M_{\alpha_1,\alpha_2}^{R}(A),$$
$$M_{r(\alpha_1,\alpha_2)}^{r(R)}[r(A)] = M_{\alpha_1,\alpha_2}^{R}(A),$$
$$M_{\alpha_1,\alpha_2}^{\lambda R}(\lambda A) = M_{\alpha_1,\alpha_2}^{R}(A), \quad (12)$$

where $r(\alpha_1, \alpha_2)$ is a simplified notation for expressing that the relative position is assessed in the direction

$$r\left(\vec{u}_{\alpha_1,\alpha_2}\right).$$

These properties are of special interest for pattern-recognition applications, since it is often required that objects should be recognized even under geometric transformations.

We have also studied the influence of the distance between objects on their relative position. We proved that when the distance between the objects increases, the objects are seen as points. The value of their relative position can be predicted only from the direction of interest and the direction in which one object goes far away from the reference object. Therefore, the shape of the objects does no longer play any role in the assessment of their relative position. See [4] for more details.

Finally, we looked at the behavior of the proposed definition on cases where the reference object has strong concavities (see [4] for more details and examples). For instance for a square inside a ring, we obtain:

$$\forall \alpha_1 \in [0, 2\pi], \prod_{\alpha_1}^{ring}(square) =$$
$$N_{\alpha_1}^{ring}(square) = M_{\alpha_1}^{ring}(square) = 1.$$

This expresses that the square is in all directions with respect to the ring. It can be interpreted as a new relative position, as "the ring surrounds the square." This relationship can be defined more generally for more complex cases as a conjunction of the degrees of relative position in all directions, as suggested in [15], e.g.,

$$\mu_{surround}^{R}(A) = \min_{\alpha_1 \in [0,2\pi]} \prod_{\alpha_1}^{R}(A),$$

or similar expressions where the minimum is taken over a given set of values of α_1. The extension to the 3D case is straightforward. Another example describes the relative position of a rectangle with respect to a strongly concave object, with increasing distance between them. The obtained results fit well the intuition concerning concave objects. In particular, it can be shown that the width of the interval $[N, \Pi]$ decreases, when the ambiguity of the relation, or the ignorance on its precise value, diminishes. Such examples illustrate an advantage of our definition, that really takes the shape of the objects into account.

4 An Illustrative Fuzzy Example on Brain Internal Structures

In this section, we illustrate the method on a fuzzy example taken from medical imaging, which shows more practical properties of the proposed approach. Other examples may be found in [2], [4], in particular on aerial images, and on 3D images. In a magnetic resonance (MR) image of the human brain we have segmented several internal structures using a fuzzy segmentation method. Five fuzzy structures are shown in Fig. 8 (with the standard "left-is-right" convention of medical images):

- left ventricle (v1),
- right ventricle (v2),
- left caudate nucleus (nc1),
- right caudate nucleus (nc2), and
- left thalamus (t1).

The fuzzy landscapes representing the degree of satisfaction of the relations "left of," "right of," "below," and "above" object v1 are shown in Fig. 9. They are obtained using (4) using the product t-norm. The relative position degrees between some of the obtained fuzzy objects are given in Fig. 10, for the t-norm min in the fuzzy pattern matching. Here again, the interpretation of these results is straightforward with respect to the intuitive expected relative positions. Object nc1 is mainly to the right of v1 (and only with very low degree to its left), and quite above and below. This expresses that it is "in the right concavity of v1," another example of more complex relationship derived from the basic relative positions. Object nc2 is to the left of v1, with no ambiguity at all concerning the right relationship (i.e., no point of nc2 is to the right of v1). It is quite above v1, and less below it than nc1. Similar interpretations can be given for t1 and v2 with respect to v1. Table 1 illustrates the symmetry property of the possibility when using the same t-norm (a product here) in the fuzzy pattern matching and in (4).

5 Conclusion

We proposed, in this paper, a new approach for defining relative position between objects in images, based on a fuzzy pattern-matching concept directly in the image space. It presents several advantages:

- it is flexible,
- it makes use of spatial fuzzy sets, directly related to the ob-

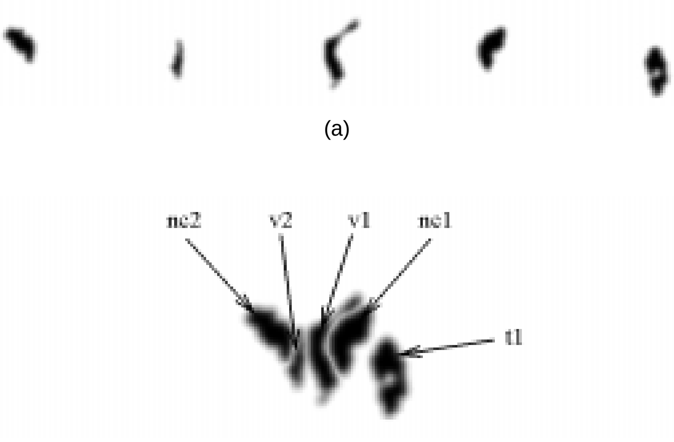


Fig. 8. (a) Five fuzzy objects resulting from a rough fuzzy segmentation of a MR brain image (membership values rank between 0 and 1, from white to black). (b) Superposition of these fuzzy objects and labels as observed in the original MR image.

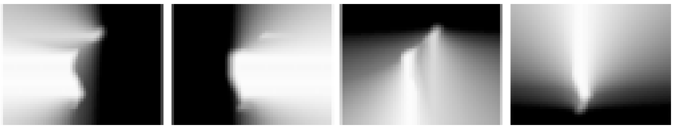


Fig. 9. Fuzzy areas corresponding to four relationships of Fig. 10 for the object v1 of Fig. 8.

jects under study,

- it takes morphological information about the shapes (2D or 3D) into account and does not rely on a reduced information like a point (dimension 0) nor on a derived function (dimension 1) like angle histogram, nor on 1D segments like histogram of forces,
- it has nice algebraic and geometrical properties,
- it is consistent with intuitive interpretation,
- it is directly applicable for 3D and fuzzy objects,
- it allows for a computation in any direction of interest, and
- it provides an evaluation as two extreme values or equivalently as an interval and an average value, which can be useful for further purposes (e.g., combination with other criteria).

The interpretation of the obtained values in terms of possibility and necessity can then be exploited in the framework of possibilistic multicriteria aggregation, as well as in the context of Dempster-Shafer evidence theory.

Foreseen applications concern spatial reasoning and structural pattern recognition. The proposed method for assessing relative position may provide a useful tool for spatial reasoning, since more complex relationships (like "between") can be derived from directional relationships, and relationships between objects can be inferred from relationships between others, using a kind of "transitivity" of relative positions. As for structural pattern recognition, we are currently developing two ways of using relative position. The first one is a relaxation labeling scheme, where we

TABLE 1
SYMMETRY OF THE DEFINED RELATIONSHIPS

Relation	Π for nc1 w.r.t. v1	Π for v1 w.r.t. nc1
left	0.220	0.992
right	0.992	0.220
below	0.761	0.826
above	0.826	0.761

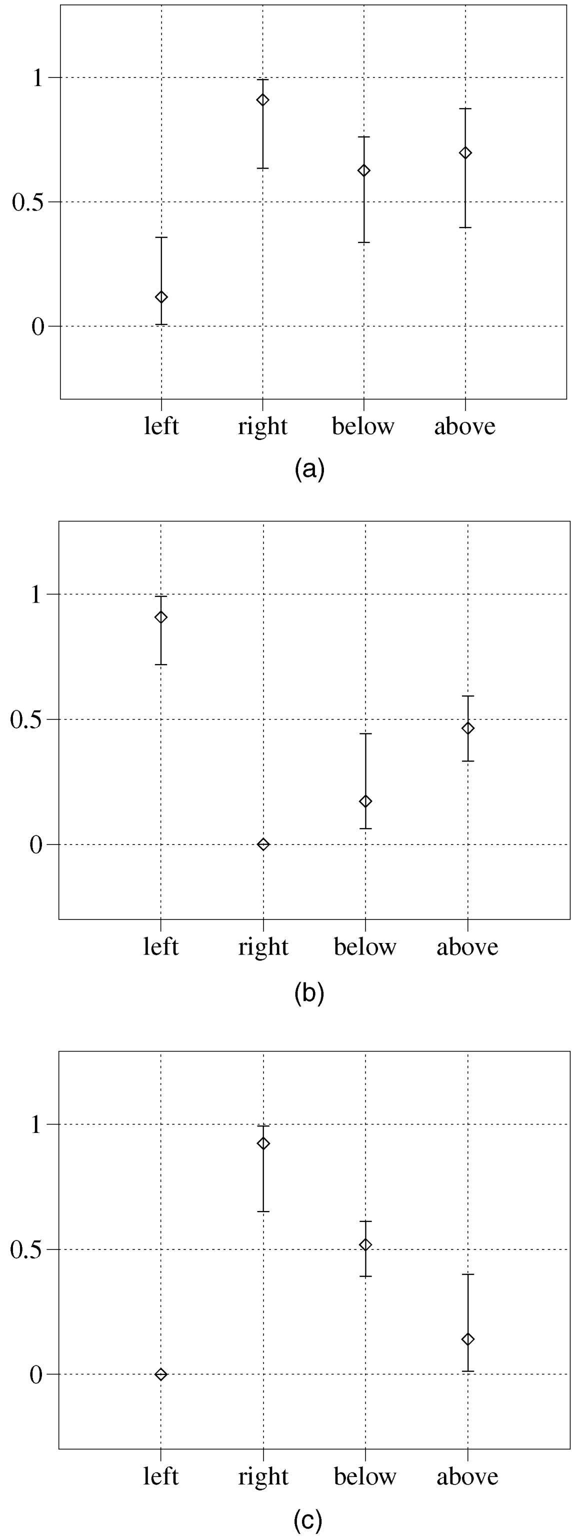


Fig. 10. Results obtained for some of the objects of Fig. 8.

try to recognize objects with respect to a model, based on relationships between these objects. The second one is a segmentation scheme in complex images, where relative position is used to restrict the search area for one object, based on a previous segmentation of another object having a known position with respect to the searched object.

REFERENCES

[1] I. Bloch, "Fuzzy Relative Position Between Objects in Images: A Morphological Approach," *Proc. ICIP '96, IEEE Int'l Conf. Image Processing*, vol. II, pp. 987–990, Lausanne, Switzerland, Sept. 1996.

[2] I. Bloch, "Fuzzy Spatial Relationships: A Few Tools for Model-Based Pattern Recognition in Aerial Images," *Proc. SPIE/EUROPTO Conf. Image and Signal Processing for Remote Sensing*, vol. 2,955, pp. 141–152, Taormina, Italy, Sept. 1996.

[3] I. Bloch, "Image Information Processing Using Fuzzy Sets," *Proc. World Automation Congress, Soft Computing with Industrial Applications*, invited conference, pp. 79–84, Montpellier, France, May 1996.

[4] I. Bloch, "Fuzzy Relative Position Between Objects in Image Processing: A Morphological Approach," Technical Report No. 97D003, ENST, Paris, 1997.

[5] I. Bloch and H. Maître, "Fuzzy Mathematical Morphologies: A Comparative Study," *Pattern Recognition*, vol. 28, no. 9, pp. 1,341–1,387, 1995.

[6] G. Borgefors, "Distance Transforms in the Square Grid," H. Maître, ed., *Progress in Picture Processing, Les Houches*, Session LVIII, 1992, ch. 1.4, pp. 46–80, North-Holland, Amsterdam, 1996.

[7] D. Dubois and H. Prade, "A Review of Fuzzy Set Aggregation Connectives," *Information Sciences*, vol. 36, pp. 85–121, 1985.

[8] D. Dubois and H. Prade, "Weighted Fuzzy Pattern Matching," *Fuzzy Sets and Systems*, vol. 28, pp. 313–331, 1988.

[9] S. Dutta, "Approximate Spatial Reasoning: Integrating Qualitative and Quantitative Constraints," *Int'l J. Approximate Reasoning*, vol. 5, pp. 307–331, 1991.

[10] J.M. Keller and X. Wang, "Comparison of Spatial Relation Definitions in Computer Vision," *Proc. ISUMA-NAFIPS '95*, pp. 679–684, College Park, Md., Sept. 1995.

[11] L.T. Koczy, "On the Description of Relative Position of Fuzzy Patterns," *Pattern Recognition Letters*, vol. 8, pp. 21–28, 1988.

[12] R. Krishnapuram and J.M. Keller, "Fuzzy Set Theoretic Approach to Computer Vision: An Overview," *Proc. IEEE Int'l Conf. Fuzzy Systems*, pp. 135–142, San Diego, 1992.

[13] R. Krishnapuram, J.M. Keller, and Y. Ma, "Quantitative Analysis of Properties and Spatial Relations of Fuzzy Image Regions," *IEEE Trans. Fuzzy Systems*, vol. 1, no. 3, pp. 222–233, 1993.

[14] P. Matsakis, L. Wendling, and J. Desachy, "A New Way to Represent the Relative Position of 2D Objects," *IEEE Trans. Pattern Analysis and Machine Intelligence*, in press.

[15] K. Miyajima and A. Ralescu, "Spatial Organization in 2D Segmented Images: Representation and Recognition of Primitive Spatial Relations," *Fuzzy Sets and Systems*, vol. 65, pp. 225–236, 1994.