Fuzzy Skeleton and Skeleton by Influence Zones: A Review

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Abstract

Skeleton is a widely addressed topic in binary image processing and object or shape representation. The difficulty raised by the discrete nature of images has led to different classes of methods. Among the most popular ones, distance based methods led to the important notion of centers of maximal balls (CMB), and mathematical morphology based methods led both to an efficient computation of CMB and to a completely different class of approaches, relying on homotopic thinning. Skeleton by influence zones (SKIZ) is another important related problem, often addressed using tools from mathematical morphology.

When imprecision has to be explicitly modeled, then objects become fuzzy sets and all the previous approaches for skeleton and SKIZ have to be extended to deal with fuzzy sets and to cope with spatial imprecision. This chapter gives an overview of the main existing definitions of fuzzy skeleton and fuzzy SKIZ, including some derived from grey level image processing, and proposes a few novel definitions.

Keywords: Fuzzy sets, fuzzy skeleton, fuzzy skeleton by influence zones, mathematical morphology, thinning, fuzzy distances, maximal disks.

1. Introduction

Representing an object by its skeleton is a widely addressed topic. It allows simplifying a shape and its description, while keeping its most relevant features. When shapes and objects are imprecisely defined, it is convenient to represent them as fuzzy sets, instead of crisp ones. This may represent different types of imprecision, either on the boundary of the objects (due for instance to partial volume effect, to the spatial resolution, or to imperfect segmentation), on the variability of these objects, on the potential ambiguity between classes, etc. In a spatial domain $\mathcal{S}$ (typically $\mathbb{R}^n$ in the continuous case or $\mathbb{Z}^n$ in the discrete case), a fuzzy set is defined via its membership function μ, a function from $\mathcal{S}$ into $[0, 1]$ providing at each point the degree to which this point belongs to the

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fuzzy set. In this chapter we will use equivalently fuzzy set and membership function. Basics on fuzzy sets can be found e.g. in [16], and a recent review of their use in image processing and understanding in [10]. Simplifying fuzzy objects then requires to extend definitions designed for the crisp case to the fuzzy case. Similarly, skeleton by influence zones (SKIZ), which structures the background of a set of objects, has to be extended when objects are imprecisely defined. In this paper, we review the main approaches for defining skeleton and skeleton by influence zones of fuzzy sets.

One of the main approaches to extend an operation to fuzzy sets is to apply this operation to α-cuts[1] and then reconstruct the result by a combination of these α-cuts. This approach is well suited for increasing operations, assuring that their application on α-cuts provides sets that can be considered as α-cuts of a resulting fuzzy set. Unfortunately skeleton and SKIZ are not increasing, and therefore this approach cannot be applied directly. Other approaches have thus been developed.

Several problems have been addressed in the literature, exploiting fuzzy representations:

- define the skeleton (as a crisp set) of a fuzzy set;
- define the fuzzy skeleton (i.e. a fuzzy set) of a fuzzy set;
- define the fuzzy SKIZ of a set of fuzzy sets.

They will be reviewed in this paper and categorized according to the type of approach they use. Note that fuzzy sets can also be used to represent some additional information on a classical skeleton. This will not be addressed here, since we focus only on fuzzy inputs, but as an example let us mention the work in [45], where a fuzzy measure of significance is computed on each branch of a classical skeleton, with applications to skeleton pruning.

Defining and computing the skeleton in the crisp case is a widely addressed topic. While there is a consensus on continuous approaches that provide the required properties (in particular representing semi-continuously a shape using thin and centered structures, respecting the topology of the initial shape, and allowing for its reconstruction)[2], several approaches have been defined in the discrete case, based on different principles, in order to address important issues directly caused by the digitization [26]. It is out of the scope of this paper to review them, and the reader may refer to [39], and recently [33], for details. The first chapters of this book also provide an overview of the field, including multi-scale aspects, and parallel implementations. The particular case of 1D skeletons of 3D objects is also reviewed in one chapter. Several applications are also described in some chapters, such as bone or vessel structure characterization. The main approaches can be grouped as follows:

[1] An α-cut of a fuzzy set μ is a crisp set defined as $\mu_\alpha = \{x \in \mathcal{S} \mid \mu(x) \geq \alpha\}$, for $\alpha \in [0, 1]$.

[2] A detailed analysis of the properties of the skeleton of an open set in $\mathbb{R}^n$ can be found in [25].

- distance-based approaches, with the important notion of center of maximal balls (CMB), and related methods such as wavefront (or grass fire) propagation, ridges of the distance function, and minimal paths;
- morphological approaches for the computation of centers of maximal balls;
- morphological approaches based on homotopic thinning.

The two first approaches correspond to the intuitive idea proposed initially by Blum [12][3], where the grass-fire principle led to the definition of the skeleton as shock points at which wavefronts intersect. These points are those that are equidistant to at least two boundary points, or equivalently the centers of maximal disks. The extensions to the fuzzy case of these two approaches are then equivalent to Blum's approach when restricted to crisp sets.

Approximate approaches in the continuous case rely on the Voronoï of points sampled on the object contour or surface. The quality of this approximation can be estimated in the case of regular objects in the sense of mathematical morphology (i.e. sets equal to their opening and closing by a disk or sphere of given radius [37]), see e.g. [14]. Regularization methods [28] and surface reconstruction methods [1] have been proposed based on this approach. In this paper, we do not consider such approximations. However the extended Voronoï of shapes will be considered for defining fuzzy SKIZ.

In the fuzzy case, we also expect the skeleton to well represent the input fuzzy shape, to be centered in the shape, to be thin enough (this notion being more complicated to define than in the crisp case). The function associating a skeleton to a fuzzy set should also be anti-extensive and idempotent, as in the crisp case. The question of homotopy is also a difficult one in the case of fuzzy sets. Finally as an extension, it is expected that the fuzzy definitions boil down to the crisp ones in the particular case where the membership functions take only values 0 and 1 (i.e. defining crisp sets).

The review proposed in this chapter is organized according to the type of approach used to define the skeleton or SKIZ of fuzzy sets, extending the above mentioned approaches. In Section 2, distance-based approaches are reviewed, with results being either crisp sets or fuzzy ones. While several approaches propose to define a crisp skeleton of a fuzzy object, fuzzy versions are interesting too, since if an object is imprecisely defined, we can expect its skeleton be imprecise too. Approaches based on mathematical morphology are presented next, for defining centers of maximal balls in Section 3, for extending the notion of thinning in Section 4, mostly based on extension of thinning to grey scale images, and finally for defining fuzzy SKIZ in Section 5. Surprisingly enough, the question of thinning of grey scale images has been addressed a long time ago, in the seventies (maybe one of the first papers is [23]), with applications to fuzzy skeleton in the early eighties. Very recently, a renewed interest for fuzzy skeleton using various approaches can be observed.

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[3] Blum's work was first presented in a conference in the 1960's, and formalized by Calabi also in the 1960's.

2. Distance-based approaches

In the crisp case, this approach leads to several almost equivalent definitions. The idea is to define a distance function inside the shape, the ridges of which build the skeleton. The skeleton points are also defined as the points through which no minimal path from any point to the shape boundary goes. According to the underlying distance on the spatial domain, maximal balls included in the shape are defined (i.e. which cannot be included in any other ball included in the shape), the centers of which building the skeleton (also called medial axis in this case). The set of centers of maximal balls is denoted $CBM(.)$ in the following. This approach can be computed efficiently according to various discrete distances [2, 13].

The ridge approach was used for fuzzy sets in [20], where the fuzzy sets were defined using a S-shape transformation of the original image. Note that this implicitly assumes that higher grey levels correspond to the expected skeleton. A ridge following algorithm was proposed, using weighted neighborhood, along with a post-processing to get a one pixel width skeleton. This approach provides a crisp result, which is thin, connected, and passes though the center of the regions with the highest membership values (note that this is not necessarily equivalent to the center of the support or of any α-cut).

The notion of maximal fuzzy disks (or more generally fuzzy balls of a distance) was proposed by several authors. In [29], the space $\mathcal{S}$ is endowed with a metric d. A fuzzy disk included in a fuzzy set μ and centered at x is defined as the following fuzzy set:

$$\forall y \in \mathcal{S}, g_x^\mu(y) = \inf_{z \mid d(x,z)=d(x,y)} \mu(z). \tag{1}$$

Note that the standard inclusion of fuzzy sets is used, expressed here as $\forall y \in \mathcal{S}, g_x^\mu(y) \leq \mu(y)$. The medial axis of μ is then defined as the set $CMB(\mu)$ of local maxima of g_x^μ (hence a crisp set), and the fuzzy medial axis transform is the set of fuzzy sets $\{g_x^\mu \mid x \in CMB(\mu)\}$.

Another definition of centers of fuzzy maximal balls was given in [24], based on the distance of a point to the thresholded fuzzy boundary of the initial fuzzy set.

An extension of the distance transform for fuzzy sets was defined in [36]. The idea is to weight the local distance between points by the membership of these points to the considered fuzzy set. For instance, the distance between two neighbor points x and y can be defined as $\max(\mu(x), \mu(y))||x - y||$, or $\frac{\mu(x)+\mu(y)}{2}||x - y||$. The length of a path is then classically defined by summing local weighted distances along the path, and the distance between two points is the length of a shortest path between these two points. Axial points, i.e. center of maximal balls, are then derived from this weigthed distance in a usual way [35]. Fuzziness is then taken into account only as weighting factors, leading to distance values defined as classical (crisp) numbers, and then a classical approach is used for the next steps. A similar approach using a discrete distance was developed in [41]. Then, based on the reserse distance transform, the centers of maximal fuzzy balls can be obtained, and define the skeleton [42]. A crisp skeleton is then obtained. A similar approach was developed in [21, 22] with further selection, filtering and refinement steps to obtain a thin crisp skeleton.

Another weighted approach was proposed in the early work published in [23], for grey level images (but could be used for fuzzy sets as well). The length of an arc is defined by a weighted sum (or integral in the continuous case). Minimum paths from each point to the contour of the support of the initial function are then computed. The skeleton is then formed by the points that do not belong to the minimum path of another point.

All these approaches are equivalent to the crisp CMB one if the input is crisp. However, the final result ignores the imprecision on the input set, providing a crisp result. One may consider that an important information is then lost, and that the result is an over-simplified representation. When transposed to the discrete case, CMB approaches suffer from the same limitations in terms of topology preservation as their crisp versions.

3. Morphological approaches to compute the centers of maximal balls

Centers of maximal balls can be obtained using mathematical morphology operations [37, 38], in particular erosion (denoted by ε) and opening (denoted by γ). The idea is that the center of a maximal ball of radius ρ is given by the set difference between the erosion of the set by a structuring element of size ρ and the opening (by the smallest possible structuring element) of this erosion. Formally, for a crisp set A, the skeleton is given by:

$$r(A) = \bigcup_{\rho>0} s_\rho(A), \tag{2}$$

where $s_\rho(A)$ is the set of centers of maximal balls of radius ρ included in A, given by:

$$s_\rho(A) = \bigcap_{\mu>0} [\varepsilon_{B_\rho}(A) \setminus \gamma_{\bar{B}_\mu}(\varepsilon_{B_\rho}(A))],$$

with B_ρ (respectively $\bar{B}_\rho$) denotes the open (respectively closed) ball of radius ρ. This definition is shown to have good properties for open sets [25]. In particular, the original set can be reconstructed from the skeleton by dilating all the s_ρ:

$$A = \bigcup_{\rho>0} \delta_{B_\rho}(s_\rho(A)).$$

In the discrete case, this definition of the skeleton is transposed as:

$$S(A) = \bigcup_{n\in\mathbb{N}} [\varepsilon_{B_n}(A) \setminus \gamma_B(\varepsilon_{B_n}(A))], \tag{3}$$

where B is the elementary structuring element on the digital grid (i.e. of radius one), and ε_{B_0} (erosion by a structuring element of radius 0) is the identity mapping. While the reconstruction property is preserved, the homotopy property is not, and we may obtain a disconnected skeleton of a connected object [26].

This type of construction was used in [30] for grey scale images, and could also be used for fuzzy sets. The idea of shrinking and expanding used in the construction of the skeleton is replaced by local min and max operators (hence corresponding to erosions and dilations). This was used in [43] for approximating an image.

Based on a similar idea, we propose to extend Equation 3 to fuzzy sets using fuzzy mathematical morphology [9, 11]. The fuzzy dilation is defined as a degree of conjunction between the translation of the (fuzzy) structuring element and the initial fuzzy set, while the fuzzy erosion is defined as a degree of implication. Opening is defined as usual as the combination of an erosion and the corresponding adjoint dilation. We propose to define the fuzzy skeleton of a fuzzy set in the discrete case as:

$$FS(\mu) = \bigvee_{n \in \mathbb{N}} [\varepsilon_{B_n}(\mu) \setminus \gamma_B(\varepsilon_{B_n}(\mu))], \tag{4}$$

where the supremum $\bigvee$ is the fuzzy union. The continuous formulation extends in a similar way. This definition applies whatever the dimension of the space, as in the crisp case. Note that instead of a crisp structuring element, a fuzzy one could be used, for instance representing the smallest spatial unit, given the imprecision in the image. An example illustrating this new definition is given in Figure 1, and shows that the skeleton is fuzzy, representing the fact that areas, even on crest lines of the membership function, may belong to the fuzzy set with a low degree, and so do the corresponding parts of the skeleton.

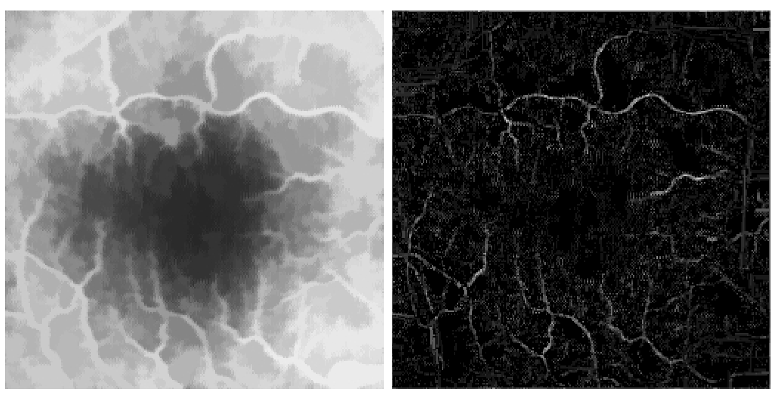


Figure 1: Left: original fuzzy set, derived from an eye vessel images. High grey leves indicate high membership degrees to the blood vessels. Right: fuzzy skeleton obtained using Equation 4, where membership degrees range from 0 (black) to 1 (white).

4. Morphological thinning

In the crisp case, a common way to overcome the homotopy problems raised by a direct computation of centers of maximal balls on digital images consists in applying iterative thinning to the input shape, while guaranteeing the preservation of topology at each step, until convergence, leading to an homotopic skeleton [37]. The structuring elements used in this process to delete so called simple points (i.e. whose deletion does not change the topology) are illustrated in Figure 2. The 3D case was also addressed, e.g. in [4, 34]. See also the corresponding chapter on 2D, 3D and 4D thinning in this book.

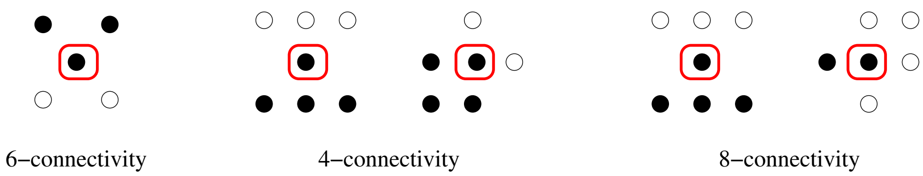


Figure 2: Structuring elements used for homotopic thinning on a hexagonal grid (6-connectivity), and on a square grid (4-connectivity or 8-connectivity). Black points represent object points while white points correspond to background points. The center of the structuring elements is circled. All rotations of these structuring elements have to be used sequentially and iteratively to get the skeleton.

This approach was extended to grey level images in several ways. Note that these extensions directly apply to fuzzy sets if grey levels are bounded and can be matched monotonically to membership degrees such that high grey levels correspond to high membership degrees (or the reverse). Historically, a first approach was introduced in [18], based on a weighted connectedness: two points x and y are connected if there exists a path from x to y that does not go through a point with higher value. Then thinning of a function f is performed by replacing $f(x)$ by $\min_{y \in V(x)} f(x)$, where $V(x)$ denotes a neighborhood of x (i.e. this corresponds to an erosion), only if this change does not disconnect any pair of points in $V(x)$.

Still among the early works, a similar approach was proposed in [19], were local min and max operators are used, referring to fuzzy logic, with structuring elements adapted to the type of thinning (for instance a 3×3 neighborhood except the center to suppress of pepper and salt noise, or oriented neighborhoods to compute the skeleton, similarly as in Figure 2).

While thinning is usually defined as the set difference between the original image and the result of a hit-or-miss transformation with appropriate structuring elements, an interesting formulation in [39] allows for a direct extension to functions. Let B_1 be the part of the structuring element composed of object points (the black points in Figure 2) and B_2 the other part of the structuring element. For thinning, the origin belongs to B_1. The thinning of a function f by $B = (B_1, B_2)$ is expressed as:

$$\forall x \in \mathcal{S}, (f \circ B)(x) = \begin{cases} \delta_{B_2}(f)(x) & \text{if } \varepsilon_{B_1}(f)(x) = f(x) \text{ and } \delta_{B_2}(f)(x) < f(x) \\ f(x) & \text{otherwise} \end{cases} \tag{5}$$

If f is a binary image, then this definition reduces to the classical one.

Another extension was proposed in [15], where topological operators for grey level images were defined. Thinning and skeleton are based on the notion of destructible point: a point x is destructible for a function f if it is simple (according to the binary case) for the threshold $f_k = \{y \mid f(y) \geq k\}$ with $k = f(x)$. Then the topology is not modified if $f(x)$ is replaced by $f(x) - 1$ (assuming that f takes values in a subset of $\mathbb{N}$). The skeleton of f is then obtained by reducing iteratively the value of destructible points that are non end-points (an end-point is a point x such that $CC(V(x) \setminus \{x\} \cap f_k) = 1$, with $k = f(x)$ and $CC(A)$ the number of connected components of A, for a given discrete connectivity).

Since thinning is often derived from hit-or-miss transformation (HMT), let us finally mention some extensions of this transformation to grey scale images. In [31], a function is expressed as the supremum of impulse functions below it, and a similar construction is used to define HMT, based on interval operators (an interval operator by (A, B) is equivalent to the HMT by (A, B^c) in the crisp case). A related approach, although somewhat different, is based on probing, as introduced in [3]. In [40], the cardinality of the set of grey levels such that a point belongs to the HMT of thresholds is considered. All these works have been nicely unified in [27] in the framework of interval operators. The idea is to decompose the HMT process into a fitting step and an evaluation step (which can be done using a sup as in [31] or a sum as in [40] or any other set measure, or using a newly proposed binary valuation). The authors show the links between the different approaches and the power of the unified framework. All definitions reduce to the classical ones if the functions are binary (i.e. sets).

5. Fuzzy skeleton of influence zones

The notions of Voronoï diagram, generalized Voronoï diagram or skeleton by influence zones (SKIZ) define regions of space which are closer to a region or object than to another one, and have important properties and applications [37, 39]. If knowledge or information is modeled using fuzzy sets, it is natural to see the influence zones of these sets as fuzzy sets too. The extension of these notions to the fuzzy case is therefore important, for applications such as partioning the space where fuzzy sets are defined, implementing the notion of separation, reasoning on fuzzy sets (fusion, interpolation, negotiations, spatial reasoning on fuzzy regions of space, etc.). Despite their interest, surprisingly enough such an extension has been little developed until now. In this section, we summarize the approach in [8], based on mathematical morphology.

In the crisp case, for a set X composed of several connected components ($X = \bigcup_i X_i$, with $X_i \cap X_j = \emptyset$ for $i \neq j$), the influence zone of X_i, denoted by $IZ(X_i)$, is defined as the set of points which are strictly closer to X_i than to X_j for $j \neq i$, according to a distance d defined on $\mathcal{S}$ (usually the Euclidean distance or a discrete version of it on digital spaces):

$$IZ(X_i) = \{x \in \mathcal{S} \mid d(x, X_i) < d(x, X \setminus X_i)\}. \quad (6)$$

The SKIZ of X, denoted by $SKIZ(X)$, is the set of points which belong to none of the influence zones, i.e. which are equidistant of at least two components X_i:

$$SKIZ(X) = (\bigcup_i IZ(X_i))^c. \tag{7}$$

The SKIZ is also called generalized Voronoï diagram. Note that the SKIZ is a subset of the morphological skeleton of X^c (i.e. the set of centers of maximal balls included in X^c where X^c denotes the complement of X in $\mathcal{S}$) [37, 39]. It is not necessarily connected and contains in general less branches than the skeleton of X^c (this may be exploited in a number of applications).

An important and interesting property of this definition based on distance (Equation 6) is that it can be expressed in terms of morphological operations as well. Let us denote by δ_λ the dilation by a ball of radius λ, and ε_λ the erosion by a ball of radius λ. Then the influence zones can be expressed as [5]:

$$IZ(X_i) = \bigcup_\lambda \left(\delta_\lambda(X_i) \cap \varepsilon_\lambda((\cup_{j \neq i} X_j)^c)\right) = \bigcup_\lambda \left(\delta_\lambda(X_i) \setminus \delta_\lambda(\cup_{j \neq i} X_j)\right). \tag{8}$$

Another link between SKIZ and distance can be expressed, by involving the watersheds (WS) [39]:

$$SKIZ(X) = WS(d(y, X), y \in X^c). \tag{9}$$

Let us now extend these definitions and properties to fuzzy sets. For the sake of clarity, we assume two fuzzy sets, with membership functions μ_1 and μ_2 defined on $\mathcal{S}$. The extension to an arbitrary number of fuzzy sets is straightforward. Fuzzy dilations and erosions are defined based on degrees of intersection (using a t-norm $\top$) and degrees of implication (using a fuzzy implication I), respectively:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} \top(\mu(y), \nu(x-y)), \tag{10}$$

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y-x), \mu(y)), \tag{11}$$

and we choose here dual definitions of these operations, with respect to a complementation c, using suitable $\top$ and I [9, 11].

5.1. Definition based on fuzzy dilations

Let us first consider the expression of influence zone using morphological dilations (Equation 8). This expression can be extended to fuzzy sets by using fuzzy intersection and union, and fuzzy mathematical morphology. For a given structuring element ν, the influence zone of μ_1 is defined as:

$$IZ_{dil}(\mu_1) = \bigcup_\lambda \left(\delta_{\lambda\nu}(\mu_1) \cap \varepsilon_{\lambda\nu}(\mu_2^c)\right) = \bigcup_\lambda \left(\delta_{\lambda\nu}(\mu_1) \setminus \delta_{\lambda\nu}(\mu_2)\right). \tag{12}$$

The influence zone for μ_2 is defined in a similar way. The extension to any number of fuzzy sets μ_i is straightforward:

$$IZ_{dil}(\mu_i) = \bigcup_\lambda \left(\delta_{\lambda\nu}(\mu_i) \cap \varepsilon_{\lambda\nu}((\cup_{j \neq i} \mu_j)^c)\right). \tag{13}$$

In these equations, intersection and union of fuzzy sets are implemented as t-norms $\top$ and t-conorms $\perp$ (min and max for instance). The fuzzy complementation used in the following is always $c(a) = 1 - a$, but other forms could be employed as well. Equation 12 then becomes:

$$IZ_{dil}(\mu_1) = \sup_{\lambda} \top[\delta_{\lambda\nu}(\mu_1), 1 - \delta_{\lambda\nu}(\mu_2)]. \tag{14}$$

In the continuous case, if ν denotes the elementary structuring element of size 1, then $\lambda\nu$ denotes the corresponding structuring element of size λ (for instance is ν is a ball of some distance of radius 1, then $\lambda\nu$ is the ball of radius λ). In the digital case, the operations performed using $\lambda\nu$ as structuring elements (λ being an integer in this case) are simply the iterations of λ operations performed with ν (iterativity property of fuzzy erosion and dilation [11]). Note that the number of dilations to be performed to compute influence zones in a digital bounded space $\mathcal{S}$ is always finite (and bounded by the length of the largest diagonal of $\mathcal{S}$).

The fuzzy SKIZ is then defined as:

$$\text{SKIZ}(\cup_i \mu_i) = (\bigcup_i IZ(\mu_i))^c. \tag{15}$$

This expression also defines a fuzzy (generalized) Voronoï diagram.

5.2. Definitions based on distances

Another approach consists in extending the definition in terms of distances (Equation 6) and defining a degree to which the distance to one of the sets is lower than the distance to the other sets. Several definitions of the distance of a point to a fuzzy set have been proposed in the literature. Some of them provide real numbers and Equation 6 can then be applied directly. But then the imprecision in the object definition is lost (the problem is the same as when using a weighted distance for computing CMB, as mentioned in Section 2). Definitions providing fuzzy numbers are therefore more interesting, since if the sets are imprecise, it may be expected that distances are imprecise too, as also underlined e.g. in [6, 17, 32]. In particular, as will be seen next, it may be interesting to use the distance proposed in [6], based on fuzzy dilation:

$$d(x, \mu)(n) = \top[\delta_{n\nu}(\mu)(x), 1 - \delta_{(n-1)\nu}(\mu)(x)]. \tag{16}$$

It expresses, in the digital case, the degree to which x is at a distance n of μ ($\top$ is a t-norm, and $n \in \mathbb{N}^*$). For $n = 0$, the degree becomes $d(x, \mu)(0) = \mu(x)$. This expression can be generalized to the continuous case as:

$$d(x, \mu)(\lambda) = \inf_{\lambda' < \lambda} \top[\delta_{\lambda\nu}(\mu)(x), 1 - \delta_{\lambda'\nu}(\mu)(x)], \tag{17}$$

where $\lambda \in \mathbb{R}^{+*}$, and $d(x, \mu)(0) = \mu(x)$.

When distances are fuzzy numbers, the fact that $d(x, \mu_1)$ is lower than $d(x, \mu_2)$ becomes a matter of degree. The degree to which this relation is satisfied can be performed using methods for comparing fuzzy numbers (see e.g. [44]). Let us consider the definition in [16], which expresses the degree $\mu(d_1 < d_2)$ to which $d_1 < d_2$, d_1 and d_2 being two fuzzy numbers, using the extension principle [46]:

$$\mu(d_1 < d_2) = \sup_{a<b} \min(d_1(a), d_2(b)). \tag{18}$$

The influence zone of μ_1 based on the comparison of fuzzy numbers (using Equation 18) is defined as:

$$\begin{aligned} IZ_{dist1}(\mu_1)(x) &= \mu(d(x,\mu_1) < d(x,\mu_2)) \\ &= \sup_{n<n'} \min[d(x,\mu_1)(n), d(x,\mu_2)(n')]. \end{aligned} \tag{19}$$

Note that this approach can be applied whatever the chosen definition of fuzzy distances.

When distances are more specifically derived from a dilation, as the ones in Equations 16 and 17, a more direct approach can be proposed, taking into account explicitly this link between distances and dilations. Indeed, in the binary case, the following equivalences hold:

$$\begin{aligned} (d(x,X_1) \leq d(x,X_2)) &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_2) \Rightarrow x \in \delta_\lambda(X_1)) \\ &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_1) \vee x \notin \delta_\lambda(X_2)). \end{aligned} \tag{20}$$

This means that if x is closer to X_1 than to X_2, x is reached faster by dilating X_1 than by dilating X_2. This expression extends to the fuzzy case as follows. The degree $\mu(d(x,\mu_1) \leq d(x,\mu_2))$ to which $d(x,\mu_1)$ is less than $d(x,\mu_2)$ is defined as:

$$\mu(d(x,\mu_1) \leq d(x,\mu_2)) = \inf_\lambda \perp(\delta_{\lambda\nu}(\mu_1)(x), 1 - \delta_{\lambda\nu}(\mu_2)(x)), \tag{21}$$

where $\perp$ is a t-conorm (fuzzy disjunction). This equation also defines a way to compare fuzzy numbers representing distances.

Defining influence zones requires a strict inequality between distances, which is derived by complementation:

$$\mu(d(x,\mu_1) < d(x,\mu_2)) = 1 - \mu(d(x,\mu_2) \leq d(x,\mu_1)). \tag{22}$$

The influence zone of μ_1 is then defined as:

$$IZ_{dist2}(\mu_1)(x) = 1 - \inf_\lambda \perp(\delta_{\lambda\nu}(\mu_2)(x), 1 - \delta_{\lambda\nu}(\mu_1)(x)). \tag{23}$$

Whatever the chosen definition of IZ, the SKIZ is always defined by Equation 15.

Comparison and properties are detailed in [8]. In particular the definitions derived from the dilation approach and from the direct distance approach are equivalent: $IZ_{dil}(\mu_1) = IZ_{dist2}(\mu_1)$. However, the two distance based approaches are not equivalent, since they rely on different orderings between fuzzy sets. Actually the direct approach always provides a larger result: $\forall x \in$

$\mathcal{S}, IZ_{dist1}(\mu_1)(x) \leq IZ_{dist2}(\mu_1)(x)$. In terms of complexity, the direct approach is computationally less expensive. Another important property is the consistency with the crisp case, as generally required when extending an operation on crisp sets to fuzzy sets. Finally, the SKIZ is symmetrical with respect to the μ_i, hence independent of their order.

5.3. Illustrative example (reproduced from [8])

The notion of fuzzy SKIZ is illustrated on the three objects of Figure 3. The structuring element ν is a crisp 3×3 square in Figure 4 and a fuzzy set of paraboloid shape in Figure 5. The influence zones of each object are displayed, as well as the SKIZ. These results are obtained with the dilation based definition. Each influence zone is characterized by high membership values close to the corresponding object, and decreasing when the distance to this object increases. The use of a fuzzy structuring element results in more fuzziness in the influence zones and SKIZ.

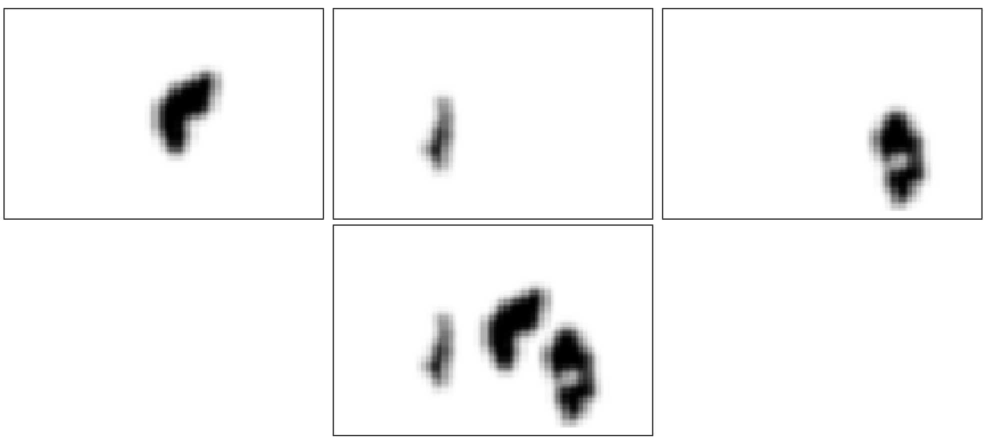


Figure 3: Three fuzzy objects and their union. Membership degrees range from 0 (white) to 1 (black).

A binary decision can be made in order to obtain a crisp SKIZ of fuzzy objects. An appropriate approach consists in computing the watershed lines of the fuzzy SKIZ: it provides spatially consistent lines, without holes, and going through the crest lines of the membership function of the SKIZ. A result is provided in Figure 6. For a fuzzy structuring element ν, the lines can go through the objects (Figure 6(b)). While this is impossible in the binary case, in the fuzzy case this is explained by the fact that an object can, to some degree, be built of several connected components, linked together by points with low membership degrees. The values of the SKIZ at those points are low too. This is the case for the third object in Figure 3. The low values of the SKIZ along the line traversing this object are in accordance with the fact that the object has only one connected component with some low degree, and two components with some higher degree. The line separating the third object can be suppressed by

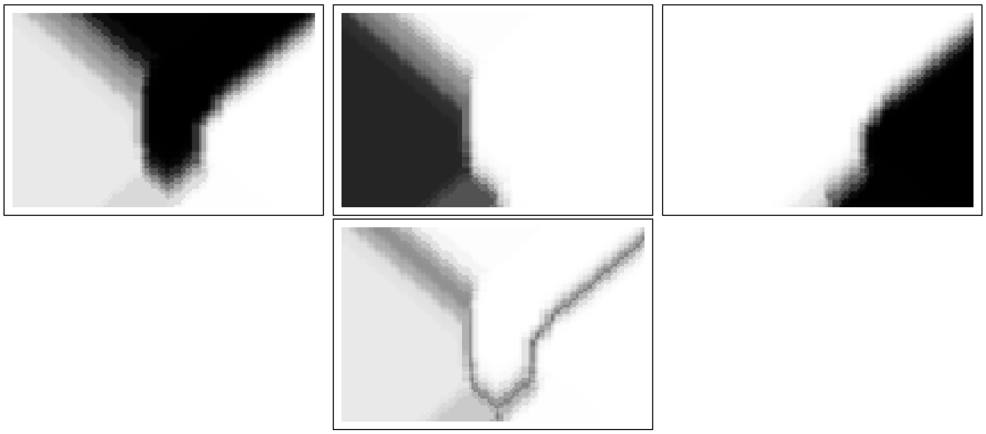


Figure 4: Influence zones of the three fuzzy objects displayed in Figure 3, and resulting fuzzy SKIZ, obtained using a binary structuring element (3×3 square) and the dilation based approach.

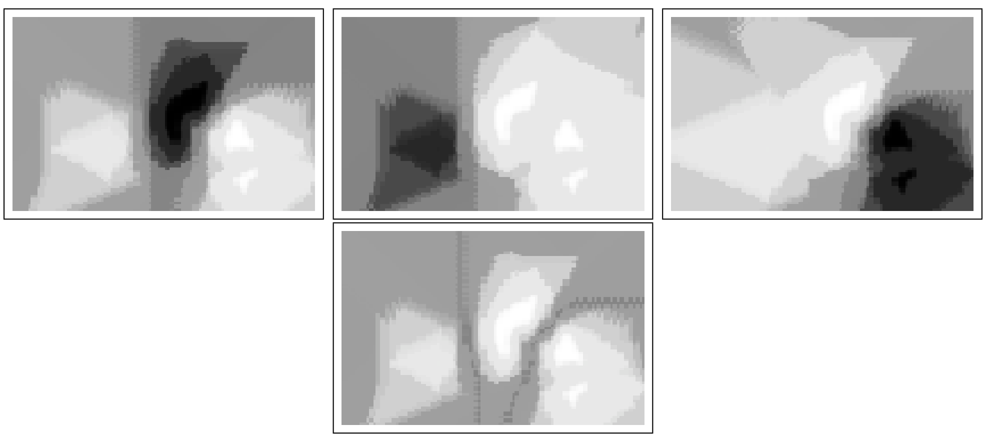
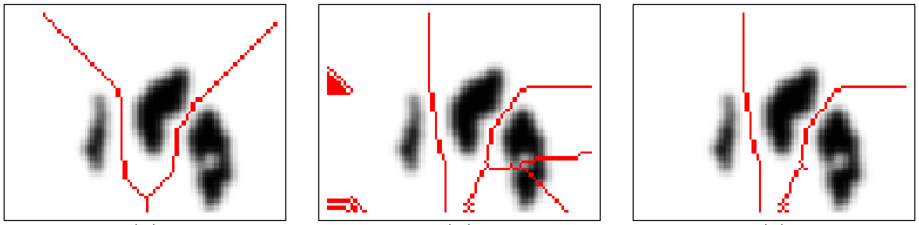


Figure 5: Influence zones of the three fuzzy objects displayed in Figure 3, and resulting fuzzy SKIZ, obtained using a fuzzy structuring element (paraboloid shaped) and the dilation based approach.

eliminating the parts of the watersheds having a very low degree in the fuzzy SKIZ (Figure 6(c)). This requires to set a threshold value.

6. Conclusion

In this chapter, the main approaches for fuzzy skeleton and fuzzy SKIZ have been reviewed. Some of them were directly designed for fuzzy sets, while others were developed for grey level images, but can be used for fuzzy sets as well, as soon as the grey level scale is bounded (and then isomorphic to $[0, 1]$). Semantics has to be considered with care to guarantee that the transformation from grey



(a) (b) (c)

Figure 6: Binary decision using watershed for ν crisp (a) and fuzzy (b). Lines with a very low membership degree in the SKIZ of (b) have been suppressed in (c).

levels to membership degrees has a suitable interpretation in terms of fuzzy sets and gradual membership.

Among these definitions, some provide crisp results, while other provide fuzzy results, which may be more convenient to keep track of the spatial imprecision of the input. This may be even more intuitive. For instance if an object has imprecise boundaries, then we would expect that points at equal distance of two or more boundary points would not be precisely located either. Similarly, thinning methods that require to define both a shape and its complement have to account for the fact that the transition is not crisply defined. Concerning semantics, it may depend on the semantics of the fuzziness in the initial object, which may represent the observation of an intrinsically imprecise object, an imprecise object representation due to sensor limitations, an object suffering from imprecision during a detection or segmentation process, a preferred region of space, etc.

Therefore a hint for future work would be to extend distance based approaches to provide fuzzy skeletons. This could be done by replacing the weighted distance transforms by distances taking values defined as fuzzy numbers (see e.g. [6, 7] for such distances).

Another aspect that deserves to be further explored is a deeper analysis of the properties of the various definitions, in particular fuzzy homotopy and its preservation (up to some degree if fuzziness if kept), measurement of how thin is a fuzzy skeleton and of how central it is in the input fuzzy set, and reconstruction capabilities. This would allow an easier comparison between different approaches, at a theoretical level.

At a more practical level, besides efficient implementations, real world applications would deserve to be more developed, to demonstrate the interest of fuzzy versions of skeleton and SKIZ.

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