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Fuzzy connectivity and mathematical morphology

Isabelle Bloch

TÉLÉCOM Paris, Département IMAGES, 46 rue Barrault, 75634 Paris Cedex 13, France

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Abstract

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We prove an equivalence between the degree of connectedness defined for fuzzy sets and the connection cost defined in the grey-level mathematical morphology framework, starting from the definitions and properties of both concepts.

Keywords. Fuzzy sets, degree of connectedness, mathematical morphology, connection cost.

Introduction

In image analysis and pattern recognition, fuzzy sets are a good representation for segmentation, classification or data fusion tasks, if the regions or classes cannot be crisply defined. They allow us for example to estimate probabilities of belonging to a class, without making a strict decision on the classification.

If the regions are represented by fuzzy sets, how should we define and measure their topological and geometrical properties and the relationships between regions? We are in this paper interested in one of these problems: the fuzzy sets connectivity, as this notion is very often involved in topological or geometrical measurements in image analysis.

In the first section, we remember how to extend the connectivity concept to fuzzy sets and give its main properties. The comparison with the connection cost defined in grey-level mathematical morphology allows us to derive in the second section and equivalence theorem between the two concepts.

Correspondence to: Dr. I. Bloch, TÉLÉCOM Paris, Département IMAGES, 46 rue Barrault, 75634 Paris Cedex 13, France.

1. Connectivity in fuzzy sets

1.1. Definitions

A fuzzy set is defined by a membership function μ, which assigns to any x belonging to the considered space E a real value $\mu(x)$ in the interval $[0,1]$ (Zadeh (1965), Kaufmann (1977)). μ extends the notion of characteristic function of the classical set theory. M denotes the set of all fuzzy sets on E. Classical sets (also named binary sets in the following) build a subset M_C of M and are characterized by functions from E to the pair $\{0,1\}$.

In order to deal with fuzzy sets, the set operations must be extended. The definitions proposed by Zadeh (1965) are the following ($\forall(\mu,\nu)\in M^2$):

intersection: $\forall x\in E,\ (\mu\cap\nu)(x)=\min(\mu(x),\nu(x))$,
union: $\forall x\in E,\ (\mu\cup\nu)(x)=\max(\mu(x),\nu(x))$,

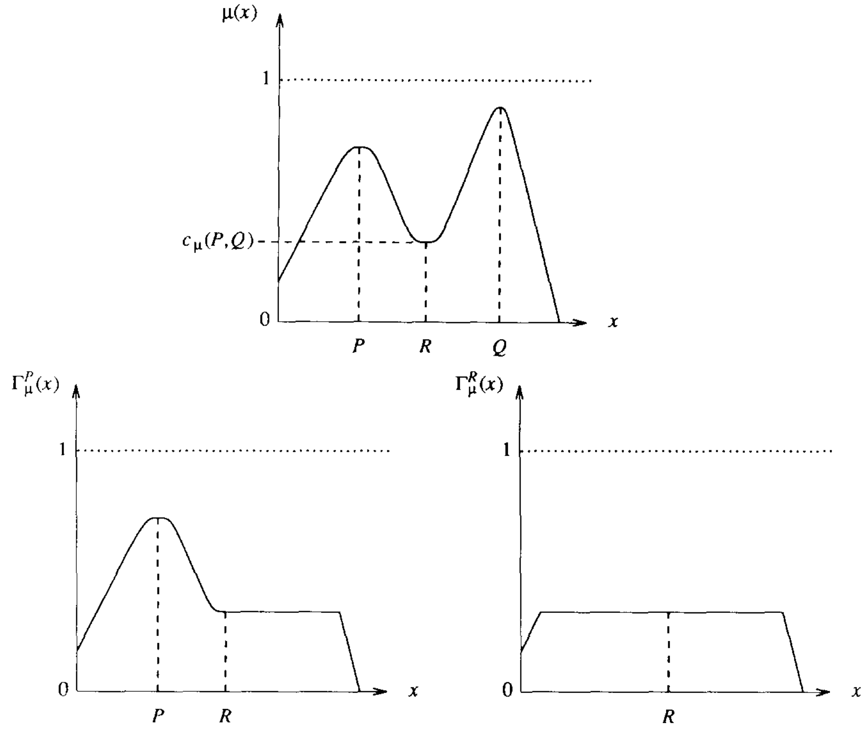

Figure 1. Degree of connectedness (above) and connected component (below) associated to P (left) and to R (right).

complement: $\forall x \in E,\ (\mu^{C})(x) = 1 - \mu(x)$,
inclusion: $\mu \subseteq \nu \Leftrightarrow \forall x \in E,\ \mu(x) \leqslant \nu(x)$.

Other definitions could be given for these operations. The previous ones present the advantage of being also true for binary sets ($\mu \in M_C$).

In a more general way, the transformations on fuzzy sets can be obtained from their corresponding set operations by means of 'fuzzification'.

Definition 1. Let Φ be a fuzzy function, from M to M. Φ is the *fuzzification* of a function Φ_C from M_C to M_C on binary sets if the restriction of Φ to M_C is equal to Φ_C:

$$\forall \mu_C \in M_C, \quad \Phi_C(\mu_C) = \Phi(\mu_C)$$

where μ_C is the characteristic function of any binary set.

Definition 2. The *threshold at level* α of a fuzzy set characterized by μ is the binary set with characteristic function μ_α ($\mu_\alpha \in M_C$):

$$\forall x \in E, \quad \mu_\alpha(x) = \begin{cases} 0 & \text{if } \mu(x) < \alpha, \\ 1 & \text{if } \mu(x) \geqslant \alpha. \end{cases}$$

A fuzzy set can be reconstructed from its thresholds by:

$$\forall x \in E, \quad \mu(x) = \int_0^1 \mu_\alpha(x)\,\mathrm{d}\alpha,$$

and the fuzzification of a binary function can be obtained in a similar manner by:

$$\Phi(\mu) = \int_0^1 \Phi_C(\mu_\alpha)\,\mathrm{d}\alpha.$$

As shown above, set operations can be extended very easily to fuzzy sets. But connectivity is much more complicated. The notion of connectivity for binary sets is generalized for fuzzy sets by means of 'degree of connectedness'. In the following, we assume that E is a discrete finite space (typically, a part of $\mathbb{Z}^2$ or $\mathbb{Z}^3$), on which a discrete connectivity is defined.

Definition 3 (Rosenfeld (1984)). The *degree of connectedness* between two arbitrary points P and Q of a fuzzy set characterized by the membership function μ is defined by:

$$c_\mu(P,Q) = \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu(P_i) \right]$$

where $L_{P,Q}=P_1 \cdots P_n$ is a path from $P=P_1$ to $Q=P_n$ in E, in the sense of the connectivity defined on E. c_μ is a function from E^2 to $[0,1]$.

Definition 4. In a fuzzy set with membership function μ, the *connected component associated with a point* Q is the fuzzy set with membership function Γ_μ^Q:

$$\forall P \in E, \quad \Gamma_\mu^Q(P) = \max_{L_{P,Q}} \left[\min_{z \in L_{P,Q}} \mu(z) \right] = c_\mu(P,Q).$$

Figure 1 illustrates the last two definitions, for a fuzzy set defined on a one-dimensional space.

1.2. Properties

In a subset A of the discrete space E, two points P and Q are called connected if and only if there exists a path $L_{P,Q}$ (for the connectivity considered on E) of points P_i of A, $1 \leqslant i \leqslant n$, such that $P=P_1$ and $Q=P_n$. This definition can also be formulated as follows:

$$\mathrm{conn}_A(P,Q) = \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu_A(P_i) \right]$$

where $L_{P,Q}$ denotes any path from P to Q and μ_A the characteristic function of the set A. P and Q are then connected in A if and only if $\mathrm{conn}_A(P,Q)=1$.

The fuzzification of the above expression leads to the following definition for the degree of connectedness in fuzzy sets:

$$\forall \mu \in M, \ \forall (P,Q) \in E^2, \quad \mathrm{conn}_\mu(P,Q) = \int_0^1 \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu_\alpha(P_i) \right] \mathrm{d}\alpha.$$

The proof of the following result is straightforward:

Proposition 1. *The degree of connectedness as defined in Definition* 3 *can be obtained through the fuzzification of the corresponding binary notion above defined, i.e.:*

$$\forall \mu \in M, \ \forall (P,Q) \in E,$$

$$c_\mu(P,Q) = \int_0^1 \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu_\alpha(P_i) \right] \mathrm{d}\alpha = \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu(P_i) \right].$$

In the binary case, the connection relationship is an equivalence relationship. The relationship properties are extended to fuzzy relationships as follows: let R be a fuzzy relationship in $[0,1]$,

- R is reflexive iff

$$\forall x \in E, \quad R(x,x) = 1,$$

- R is symmetrical iff

$$\forall (x,y) \in E^2, \quad R(x,y) = R(y,x),$$

- R is transitive iff $\forall (x,z) \in E^2$,

$$R(x,z) \geqslant \max_{y \in E} [\min(R(x,y), R(y,z)].$$

Proposition 2. *Thus the properties of the degree of connectedness are the following:*

- *c_μ is symmetrical,*
- *c_μ is transitive,*
- *$c_\mu(x,x)=\mu(x)$, thus c_μ is not reflexive, but only weakly reflexive ($\forall y \in E, c_\mu(x,x) \geqslant c_\mu(x,y)$, see Kaufmann* (1977)).[1]

In non-mathematical terms, the degree of connectedness is obtained by walking from P to Q and 'descending as little as possible' in the membership values along the path. It is strongly related to the membership values of P and Q, and in particular, we have:

$$\forall \mu \in M, \ \forall (P,Q) \in E^2, \quad c_\mu(P,Q) \leqslant \min(\mu(P), \mu(Q)).$$

Thus, a point with low membership to the fuzzy set

[1] Other definitions are possible for the degree of connectedness. For example, the following definition gives a reflexive, symmetrical, non-transitive relationship, and two points belonging to a plateau have a degree of connectedness of 1, independently of the height of the plateau:

$$c_\mu^{(2)}(P,Q) = 1 - [\min(\mu(P), \mu(Q)) - \max_{L_{P,Q}} \left[\min_{1 \leqslant i \leqslant n} \mu(P_i)) \right].$$

The two definitions are compared by Zachmann (1990). The definition for c_μ here adopted is the one which generally provides the best properties for pattern recognition.

also has a low degree of connectedness to any other point. Moreover, two points on a plateau at height α have a degree of connectedness equal to α.

2. Degree of connectedness and morphological connection cost

2.1. Connection cost in mathematical morphology

This section is set in the framework of grey-level mathematical morphology, for functions defined from $\mathbb{R}^n$ to $\mathbb{R}$, upper-semicontinuous, with connected support, and upper-bounded over any bounded subset of their support (Serra (1982)).

The connection cost is defined in this framework from the geodesic distance calculated in the thresholds of the considered function.

Definition 5. The *threshold of a function f at level* α is the (binary) set X_α defined by:

$$X_\alpha = \{x \in \mathrm{supp}(f), f(x) \leqslant \alpha\}$$

where $\mathrm{supp}(f)$ denotes the support of the function f.

Definition 6. The *geodesic distance* δ_X *with respect to a set* X is defined for two points in X as the length of the shortest path between the two points which is included in X, if such a path exists, and $+\infty$ else. If one (at least) of the two points does not belong to X, the geodesic distance is equal to 0 if the points are the same, to $+\infty$ else.

Definition 7 (Prêteux and Merlet (1991)). The *connection cost related to a function f* is defined for $x \neq y$ by:

$$\xi_f(x,y) = \inf\{\alpha \in \mathbb{R}, \delta_{X_\alpha}(x,y) < +\infty\}$$

and, by convention, $\xi_f(x,x) = -\infty$.

Proposition 3 (Prêteux and Merlet (1991)). *The connection cost can be expressed from the function values of f along all paths* $L_{x,y}$ *from x to y by*:

$$\forall (x,y) \in (\mathrm{supp}(f))^2,\ x \neq y,$$
$$\xi_f(x,y) = \inf_{L_{x,y}} \left[\sup_{z \in L_{x,y}} f(z) \right].$$

Proposition 4 (Prêteux and Merlet (1991)). *The connection cost has the following properties*:

$$\forall (x,y) \in (\mathrm{supp}(f))^2, \quad \xi_f(x,y) = \xi_f(y,x),$$
$$\forall (x,y,z) \in (\mathrm{supp}(f))^3,$$
$$\xi_f(x,z) \leqslant \max(\xi_f(x,y), \xi_f(y,z)),$$
$$\forall (x,y) \in (\mathrm{supp}(f))^2,\ x \neq y,$$
$$\xi_f(x,y) \geqslant \max(f(x), f(y)).$$

2.2. An equivalence theorem

The equation in Proposition 3 and the properties in Proposition 4 are of the same type as those of the degree of connectedness for fuzzy sets (see Section 1). The following theorem gives an equivalence between these two concepts.

Theorem. *Let us consider functions defined on a discrete finite grid, bounded, taking their values in* $\mathbb{R}^+$ *(we can assume, without loss of generality, that they are taking their values in* $[0,1]$ *and thus they can be seen as membership functions of fuzzy sets), and set for such a function* μ *the convention* $\xi_\mu(x,x) = \mu(x)$. *Then we have*:

$$\forall \mu \in M,\ \forall (P,Q) \in E^2,$$
$$c_{(1-\mu)}(P,Q) = 1 - \xi_\mu(P,Q).$$

This equation is suggested by the following observation on the analogy between the thresholds at level α defined respectively for fuzzy sets and in grey-level mathematical morphology:

Lemma. *With notations identical to previous ones, we have*:

$$\forall \mu \in M,\ \forall \alpha \in [0,1], \quad (1-\mu)_{(1-\alpha)} = \mu_{X_\alpha}$$

where μ_{X_α} *is the characteristic function of the binary set* X_α.

Proof of Lemma. From the definition of the thresholds of a fuzzy set, we derive, for any $\mu \in M$, $\alpha \in [0,1]$, and $x \in E$:

$$(1-\mu)_{(1-\alpha)}(x) = 0 \ \Leftrightarrow\ 1-\mu(x) < 1-\alpha$$
$$\Leftrightarrow\ \mu(x) > \alpha$$

and

$$(1-\mu)_{(1-\alpha)}(x) = 1 \;\Leftrightarrow\; 1-\mu(x) \geqslant 1-\alpha$$
$$\Leftrightarrow\; \mu(x) \leqslant \alpha.$$

From the definition of the thresholds in mathematical morphology, we derive:

$$\mu_{X_\alpha}(x) = 0 \;\Leftrightarrow\; \mu(x) > \alpha$$

and

$$\mu_{X_\alpha}(x) = 1 \;\Leftrightarrow\; \mu(x) \leqslant \alpha.$$

Providing the desired equality. □

Proof of Theorem. When working on a discrete grid, 'sup' and 'inf' become 'max' and 'min' (bounds are reached). From the definition of connection cost, we have:

$$\exists L_{P,Q},\ \forall P_i \in L_{P,Q},\quad \mu(P_i) \leqslant \xi_\mu(P,Q)$$

and

$$\forall \alpha < \xi_\mu(P,Q),\ \forall L_{P,Q},\ \exists P_i \in L_{P,Q},\quad \mu(P_i) > \alpha.$$

The above two equations are equivalent to:

$$\exists L_{P,Q},\ \forall P_i \in L_{P,Q},\quad 1-\mu(P_i) \geqslant 1-\xi_\mu(P,Q)$$

and

$$\forall \alpha < \xi_\mu(P,Q) (\text{then } 1-\alpha > 1-\xi_\mu(P,Q)),$$
$$\forall L_{P,Q},\ \exists P_i \in L_{P,Q},\quad 1-\mu(P_i) < 1-\alpha.$$

Setting $\beta_0 = 1-\xi_\mu(P,Q)$ and $\beta = 1-\alpha$, the two previous equations are equivalent to:

$$\beta_0 = \max\{\beta \mid \exists L_{P,Q},\ \forall P_i \in L_{P,Q}, 1-\mu(P_i) \geqslant \beta\}.$$

Thus, β_0 is exactly the degree of connectedness between P and Q in the fuzzy set defined by the membership function $1-\mu$, and

$$c_{(1-\mu)}(P,Q) = 1-\xi_\mu(P,Q). \qquad \square$$

Figure 2 illustrates the equivalence between connection cost and degree of connectedness.

2.3. Interpretation

This equivalence allows us to relate connectivity notions defined for fuzzy sets with topographical notions derived directly from the connection cost. It allows us to combine two completely different formalisms, the one dedicated to decision making (among others), and the other to morphological analysis: fuzzy sets allow us for instance to solve classification problems and connectivity appears in this domain as a useful tool to introduce context, or coherence; on the other hand, grey-level mathematical morphology was based on neighbourhood operations (in contrast, binary mathematical morphology uses distance notions) and the connection cost now allows us to define a topographical distance from which a topographical morphology is derived.

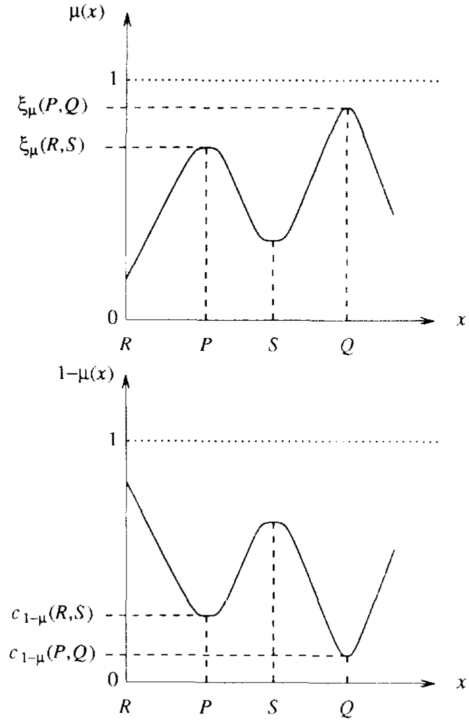

Figure 2. Connection cost and degree of connectedness.

3. Conclusion

A new association has been presented between fuzzy sets and mathematical morphology by proving an equivalence between two concepts: degree of connectedness for fuzzy sets, and connection cost for grey-level mathematical morphology. Thus, the two formalisms inherit directly the properties, transformations and applications related to and derived from these two notions.

References

Kaufmann, A. (1977). *Introduction à la théorie des sous-ensembles flous.* Masson, Paris.

Prêteux, F. and Merlet, N. (1991). New concepts in mathematical morphology: the topographical and differential distance functions. *SPIE* 1568, *Image Algebra and Morphological Image Processing II*, 66–77.

Rosenfeld, A. (1984). The fuzzy geometry of image subsets. *Pattern Recognition Lett.* 2, 311–317.

Serra, J. (1982). *Image Analysis and Mathematical Morphology.* Academic Press, London.

Zachmann, H.H. (1990). Untersuchungen zur automatischen Interpretation von kranialen Kernspintomogrammen mit Hilfe einer ikonischen Wissensbasis. Diplomarbeit, Universität Hamburg, Fachbereich Informatik.

Zadeh, L.A. (1965). Fuzzy sets. *Inform. Control* 8, 338–353.