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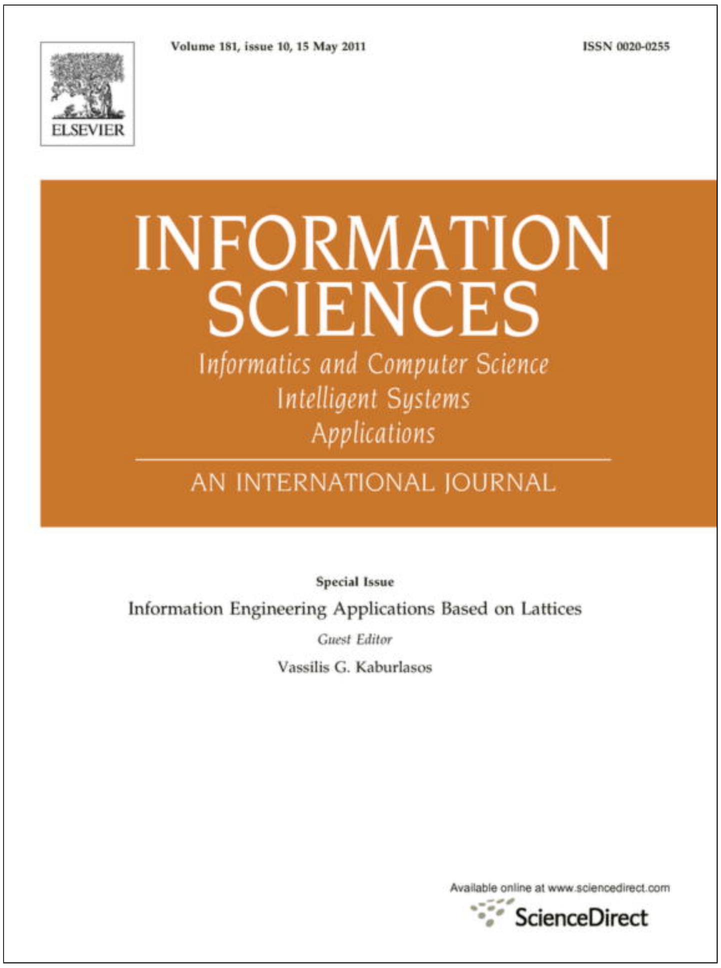

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Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology

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A R T I C L E I N F O

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A B S T R A C T

Mathematical morphology is based on the algebraic framework of complete lattices and adjunctions, which endows it with strong properties and allows for multiple extensions. In particular, extensions to fuzzy sets of the main morphological operators, such as dilation and erosion, can be done while preserving all properties of these operators. Another extension concerns bipolar fuzzy sets, where both positive information and negative information are handled, along with their imprecision. We detail these extensions from the point of view of the underlying lattice structure. In the case of bipolarity, its two-components nature raises the question of defining a proper partial ordering. In this paper, we consider Pareto (component-wise) and lexicographic orderings.

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1. Introduction

Lattice theory has become a popular mathematical framework in different domains of information processing, such as mathematical morphology, fuzzy sets, formal concept analysis, among others. Here we consider two important extensions of mathematical morphology, from the point of view of lattice theory. The first one deals with imprecision and vagueness in knowledge representation and information processing, in the framework of fuzzy sets. The second one deals additionally with the bipolarity of information, which occurs in several domains, such as preference modeling under some constraints, spatial reasoning, argumentation, etc. In these domains, two types of information have often to be handled: (i) positive information (e.g. what is possible or desired), and (ii) negative information (e.g. rules, constraints). Extending mathematical morphology to these frameworks has applications in many domains, including image processing, spatial reasoning, logics, negotiations [22,14,12,27].

In Section 2, we briefly summarize the algebraic framework of mathematical morphology, for the sake of completeness. The case where the underlying space is the spatial domain and where operators are subject to an additional property of invariance under translation is specified, since it is important for many applications (image processing, computer vision, spatial reasoning,...). The extension to fuzzy sets is described in Section 3, summarizing our previous work. A more recent and novel contribution concerns the extension to bipolar fuzzy sets. Since we have to handle two components (positive and negative information), the question of the partial ordering is raised. We detail here the case of the Pareto partial ordering, which operates component-wise, in Section 4, extending some preliminary work, and the case of the lexicographic ordering in Section 5, which is a completely new contribution of this paper. These orderings are but two examples of possible orderings, representing "extreme" situations, where the positive and negative components are handled in the same way (Pareto ordering), or in contrary where a strong priority is given to one of them (lexicographic ordering). In Section 6 we mention other possible orderings, opening new perspectives to this work.

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2. Algebraic framework of mathematical morphology

Mathematical morphology [70] usually relies on the algebraic framework of complete lattices [67]. However, it has also been extended to complete semi-lattices and general posets [53], based on the notion of adjunction [50] (see also [22] for a general description of the algebraic framework). Here, within the scope of this special issue, we only consider the case of complete lattices.

Let $(\mathcal{T}, \leqslant)$ be a complete lattice, where $\leqslant$ is a partial ordering on $\mathcal{T}$, and let $\vee$ be the supremum and $\wedge$ the infimum. A dilation is an operator δ on $\mathcal{T}$ which commutes with the supremum and an erosion is an operator ε on $\mathcal{T}$ which commutes with the infimum [50]:

$$\forall (x_i) \in \mathcal{T}, \quad \delta(\vee_i x_i) = \vee_i \delta(x_i), \quad \varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i). \tag{1}$$

Such operators are called algebraic dilations and erosions. An important property is that they are increasing with respect to $\leqslant$.

An adjunction on $(\mathcal{T}, \leqslant)$ is a pair of operators (ε, δ) such that:

$$\forall (x, y) \in \mathcal{T}^2, \quad \delta(x) \leqslant y \Longleftrightarrow x \leqslant \varepsilon(y). \tag{2}$$

A major property links adjunctions and algebraic morphological operators: if (ε, δ) is an adjunction, then ε is an algebraic erosion and δ an algebraic dilation. Additionally, the following properties hold:

- if (ε, δ) is an adjunction, then δ preserves the smallest element of $\mathcal{T}$ et ε preserves the largest element;
- if (ε, δ) is an adjunction, then $\varepsilon\delta \geqslant Id$, where Id denotes the identity mapping on $\mathcal{T}$, and $\delta\varepsilon \leqslant Id$ (the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are known as morphological opening and closing, respectively, and can also be formalized in the framework of Moore families [68]);
- if (ε, δ) is an adjunction, then $\varepsilon\delta\varepsilon = \varepsilon,\ \delta\varepsilon\delta = \delta,\ \varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$, i.e. the compositions $\varepsilon\delta$ and $\delta\varepsilon$ are idempotent;
- let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive (i.e. $\delta\varepsilon \leqslant Id$) and $\varepsilon\delta$ is extensive (i.e. $\varepsilon\delta \geqslant Id$). Then (ε, δ) is an adjunction;
- if ε is an increasing operator, it is an algebraic erosion if and only if there exists δ such that (ε, δ) is an adjunction. The operator δ is then an algebraic dilation and can be expressed as: $\delta(x) = \bigwedge\{y \in \mathcal{T} \mid x \leqslant \varepsilon(y)\}$ (representation theorem). A similar representation result holds for erosion.

Let us now consider $\mathbb{R}^n$ or $\mathbb{Z}^n$, representing the spatial domain denoted by $\mathcal{S}$ in the following. In the particular case of the lattice of subparts of $\mathcal{S}$, endowed with inclusion as partial inclusion, adding a property of invariance under translation leads to the particular following forms (called morphological dilations and erosions):

$$\forall X \subseteq \mathcal{S}, \quad \delta_B(X) = \{x \in \mathcal{S} \mid \check{B}_x \cap X \neq \emptyset\}, \quad \varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}, \tag{3}$$

where B is a subset of $\mathcal{S}$ called structuring element, B_x denotes its translation at point x and $\check{B}$ its symmetrical with respect to the origin of space. Opening and closing are defined by composition (using the same structuring element). These are the usual forms of the basic mathematical morphology operators, used typically in image processing and in spatial reasoning.

Note that B can be considered in a more general way as a binary relation between two points of $\mathcal{S}$ (i.e. y is in relation with x if and only if $y \in B_x$). This allows establishing interesting links with several other domains, such as rough sets [11], and, in the more general case where the morphological operations are defined from one set to another one, with Galois connections and formal concept analysis, as shown e.g. in [22].

These definitions are general and apply to any complete lattice. In the following, we focus on the lattice of fuzzy sets defined on $\mathcal{S}$ and on the lattice of bipolar fuzzy sets. Other works, not described in this paper, have been done on the lattice of logical formulas in propositional logics [23,25–27], with applications to fusion, revision, abduction, mediation, or in modal logics [12], with applications including qualitative spatial reasoning. Mathematical morphology can therefore be considered as a unifying framework for spatial reasoning, leading to knowledge representation models and reasoning tools in quantitative, semi-quantitative (or fuzzy) and qualitative settings [22,14].

3. Fuzzy mathematical morphology

Extending mathematical morphology to fuzzy sets was proposed in the early 90's, by several teams independently [7,10,36,37,71], and was then largely developed (see e.g. [24,35,38,57,59,65]). An earlier extension of Minkowski's addition (which is directly linked to dilation) was defined in [45].

A fuzzy set on $\mathcal{S}$ is defined through its membership function μ from $\mathcal{S}$ into $[0,1]$, where $\mu(x)$ represents the degree to which point x of $\mathcal{S}$ belongs to the fuzzy set. Let $\mathcal{F}$ be the set of fuzzy subsets of $\mathcal{S}$. For the sake of simplicity, we identify a fuzzy set with its membership function. An usual partial ordering $\leqslant$ on $\mathcal{F}$ is defined as: $\mu \leqslant \nu \Longleftrightarrow \forall x \in \mathcal{S},\ \mu(x) \leqslant \nu(x)$, and $(\mathcal{F}, \leqslant)$ is a complete lattice. The smallest element is the function constantly equal to 0 and the largest element the function constantly equal to 1. Infimum and supremum are the point-wise min and max, respectively, denoted by $\wedge$ and $\vee$ when they apply to any family of fuzzy sets.

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3.1. A brief reminder on fuzzy connectives

For the sake of completeness, we recall in this section a few definitions of fuzzy connectives, used in the following (see e.g. [44] for more details).

A fuzzy implication $\mathcal{I}$ is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is decreasing in the first argument, increasing in the second one and satisfies $\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1$ and $\mathcal{I}(1,0) = 0$.

A fuzzy conjunction T is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $T(0,0) = T(1,0) = T(0,1) = 0$ and $T(1,1) = 1$. If T is also associative and commutative and satisfies $\forall \alpha \in [0,1],\ T(\alpha,1) = T(1,\alpha) = \alpha$, it is a t-norm, and this is the form of conjunction that will be used in the following.

A fuzzy disjunction T^C is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $T^C(1,1) = T^C(1,0) = T^C(0,1) = 1$ and $T^C(0,0) = 0$. If T^C is also associative and commutative and satisfies $\forall \alpha \in [0,1],\ T^C(\alpha,0) = T^C(0,\alpha) = \alpha$, it is a t-conorm.

A negation (or complementation in a set theoretical terminology) is a mapping c from $[0,1]$ into $[0,1]$, which is decreasing and satisfies $c(1) = 0$ and $c(0) = 1$. In this paper, we consider only involutive negations, i.e. such that $\forall \alpha \in [0,\,1], c(c(\alpha)) = \alpha$, the interest of which being highlighted e.g. in [56].

Typical examples are [44]:

- mininum $\min(a,b)$, product ab, or Lukasiewicz $\max(0,a+b-1)$ operators for t-norms,
- maximum $\max(a,b)$, algebraic sum $a+b-ab$, or Lukasiewicz $\min(1,a+b)$ operators for t-conorms,
- and $c(\alpha) = 1-\alpha$ for negation.

These connectives can be linked to each other in different ways. In particular, using duality and residuation principles. The following examples will be used in this paper:

- a t-conorm can be derived from a t-norm and a negation as: $\forall(\alpha,\beta) \in [0,1]^2,\ T^C(\alpha,\beta) = c(T(c(\alpha),c(\beta)))$, and is then called the dual t-conorm of T;
- a common way to define an implication $\mathcal{I}$ (called S-implication), consists in deriving it from a t-conorm T^C and a negation c [44]:

$$\forall(\alpha,\beta) \in [0,1]^2, \quad \mathcal{I}(\alpha,\beta) = T^C(c(\alpha),\beta), \tag{4}$$

this equation translating directly the crisp logical equivalence between $(\varphi \Rightarrow \psi)$ and $(\psi \vee \neg\varphi)$;
- another interesting and usual approach to derive an implication $\mathcal{I}$ (called R-implication) is to apply the residuation principle, from a t-norm T:

$$\forall(\alpha,\beta) \in [0,1]^2, \quad \mathcal{I}(\alpha,\beta) = \sup\{\gamma \in [0,1] \,|\, T(\alpha,\gamma) \leqslant \beta\}. \tag{5}$$

This definition coincides with the previous one for particular forms of T, typically the Lukasiewicz t-norm.

3.2. Algebraic fuzzy dilation and erosion

Algebraic operations can be defined on $(\mathcal{F}, \leqslant)$, as described in Section 2.

Definition 1. A dilation on $(\mathcal{F}, \leqslant)$ is an operator that commutes with the supremum and an erosion is an operator that commutes with the infimum:

$$\forall(\mu_i) \in \mathcal{F}, \quad \delta(\vee_i \mu_i) = \vee_i \delta(\mu_i), \quad \varepsilon(\wedge_i \mu_i) = \wedge_i \varepsilon(\mu_i). \tag{6}$$

All properties derived from the general algebraic framework, as described in Section 2, then hold.

3.3. Morphological fuzzy dilation and erosion, using a fuzzy structuring element

Adding a property of invariance under translation leads to the following general forms of fuzzy dilations and erosions [10,24] (we actually proved that these forms are the most general ones in order to fulfill all properties of classical mathematical morphology [19]):

Definition 2. Let μ be a fuzzy set and ν a fuzzy structuring element (an element of $\mathcal{F}$). The morphological dilation and erosion of μ by ν are defined as:

$$\forall x \in \mathcal{S}, \quad \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} T[\nu(x-y), \mu(y)], \quad \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} \mathcal{I}[\nu(y-x), \mu(y)], \tag{7}$$

where T is a t-norm and $\mathcal{I}$ a fuzzy implication.[1]

[1] Note that this set theoretical view, extended to the fuzzy case, differs from gray-scale morphology, where the sum would be used instead of T [24].

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The value at point x of the dilation of μ by ν represents the degree to which the translation of the structuring element ν at point x intersects μ, while the erosion value at x represents the degree to which it is included in μ.

The adjunction property imposes that $\mathcal{I}$ be the residual implication of T (see Eq. (5)).

The t-norm T defines a conjunction. A disjunction can be defined from an implication and a negation c (which has to be involutive in the context of mathematical morphology) as: $\forall(\alpha,\beta)\in[0,1]^2,\ T^C(\alpha,\beta)=\mathcal{I}(c(\alpha),\beta)$ (see Eq. (4)). For applications dealing with spatial objects for instance, it is often important to also have a duality property between dilation and erosion, with respect to the complementation. Then T and T^C have to be dual operators with respect to c. This property, along with the adjunction property, limits the choice of T and T^C to generalized Lukasiewicz operators [13,19]: $T(\alpha,\beta)=\varphi^{-1}(\max(0,\varphi(\alpha)+\varphi(\beta)-1))$ and $T^C(\alpha,\beta)=\varphi^{-1}(\min(1,\varphi(\alpha)+\varphi(\beta)))$ where φ is a continuous strictly increasing function on [0,1] with $\varphi(0)=0$ and $\varphi(1)=1$. For these operators, all properties of classical morphology do hold.

The links between definitions obtained for various forms of conjunctions and disjunctions have been presented from different perspectives in [24,59,13,19,72].

Opening and closing are defined by composition, as in the general case. The adjunction property guarantees that these operators are idempotent, and that opening (respectively closing) is anti-extensive (respectively extensive) [24,35,19].

4. Bipolar fuzzy mathematical morphology and Pareto ordering

Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [43]. This domain has recently motivated work in several directions, for instance for applications in knowledge representation, preference modeling, argumentation, multi-criteria decision analysis, cooperative games, among others [1,9,28,31,47,49,52,54,63,64,66]. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [43,46,8,9]. In this paper, we propose to define morphological operators on this type of representation.

A bipolar fuzzy set on $\mathcal{S}$ is defined by a pair of functions (μ,ν) such that $\forall x\in\mathcal{S},\ \mu(x)+\nu(x)\leqslant 1$. For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ its non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. The constraint on the two functions expresses a consistency between positive and negative information [43]. Since the positive information models what is possible, preferred, observed or experienced, and the negative information what is forbidden or impossible, the consistency constraint avoids contradictions between what is forbidden and what is possible (i.e. the potential solutions should be included in what is not forbidden or impossible). Note that fuzzy sets are then particular cases of bipolar fuzzy sets, when $\nu=1-\mu$.

4.1. Pareto ordering and derived lattice

Let us consider the set $\mathcal{L}$ of pairs of numbers (a,b) in [0,1] such that $a+b\leqslant 1$. It is a complete lattice, for the partial order defined as [34]: $(a_1,b_1)\preceq(a_2,b_2)$ iff $a_1\leqslant a_2$ and $b_1\geqslant b_2$. This corresponds to a component-wise partial ordering, which is equivalent to Pareto ordering (often used in economics and social choice) by reversing the scale of negative information. The greatest element is (1,0) and the smallest element is (0,1). The supremum and infimum are respectively defined as: $(a_1,b_1)\vee(a_2,b_2)=(\max(a_1,a_2),\min(b_1,b_2)),\ (a_1,b_1)\wedge(a_2,b_2)=(\min(a_1,a_2),\max(b_1,b_2))$. The partial order $\preceq$ induces a partial order on the set of bipolar fuzzy sets defined on $\mathcal{S}$, denoted by $\mathcal{B}$.

Definition 3. A Pareto ordering on $\mathcal{B}$ is defined as:

$\forall(\mu_1,\nu_1)\in\mathcal{B},\ \forall(\mu_2,\nu_2)\in\mathcal{B}$

$$(\mu_1,\nu_1)\preceq(\mu_2,\nu_2) \text{ iff } \forall x\in\mathcal{S},\ \mu_1(x)\leqslant\mu_2(x) \text{ and } \nu_1(x)\geqslant\nu_2(x). \tag{8}$$

In all what follows, for each $(\mu,\nu)\in\mathcal{B}$, we will note $(\mu,\nu)(x)=(\mu(x),\nu(x))\ (\in\mathcal{L}),\ \forall x\in\mathcal{S}$.

Proposition 1. *$(\mathcal{B},\preceq)$ is a complete lattice. The supremum and the infimum are:*

$$\forall x\in\mathcal{S},((\mu_1,\nu_1)\vee(\mu_2,\nu_2))(x)=(\max(\mu_1(x),\mu_2(x)),\min(\nu_1(x),\nu_2(x))), \tag{9}$$

$$\forall x\in\mathcal{S},((\mu_1,\nu_1)\wedge(\mu_2,\nu_2))(x)=(\min(\mu_1(x),\mu_2(x)),\max(\nu_1(x),\nu_2(x))). \tag{10}$$

Let us now consider any family of bipolar fuzzy sets $(\mu_i,\nu_i),\ i\in I$, where the index set I can be finite or not. We have similar expressions of supremum and infimum:

$$\forall x\in\mathcal{S},\bigvee_{i\in I}(\mu_i,\nu_i)(x)=\left(\sup_{i\in I}\mu_i(x),\inf_{i\in I}\nu_i(x)\right),$$

$$\forall x\in\mathcal{S},\bigwedge_{i\in I}(\mu_i,\nu_i)(x)=\left(\inf_{i\in I}\mu_i(x),\sup_{i\in I}\nu_i(x)\right),$$

The greatest element is the pair of functions constantly equal to 1 and to 0 respectively, and the smallest element is the pair of functions constantly equal to 0 and to 1 respectively.

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4.2. Algebraic operators

Mathematical morphology on bipolar fuzzy sets has been first introduced in [15]. Once we have a complete lattice, it is easy to define algebraic dilations and erosions on this lattice, as described in Section 2.

Definition 4. A dilation is an operator δ from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the supremum:

$$\forall(\mu_i, \nu_i) \in \mathcal{B}, \quad \delta(\vee_i(\mu_i, \nu_i)) = \vee_i \delta((\mu_i, \nu_i)). \tag{11}$$

An erosion is an operator ε from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the infimum:

$$\forall(\mu_i, \nu_i) \in \mathcal{B}, \quad \varepsilon(\wedge_i(\mu_i, \nu_i)) = \wedge_i \varepsilon((\mu_i, \nu_i)). \tag{12}$$

Their properties are directly derived from the lattice framework, and are exactly the same as the ones described in Section 2.

4.3. Morphological operators

We now assume that $\mathcal{S}$ is an affine space (or at least a space on which translations can be defined). The general principle underlying morphological erosions (respectively dilations) consists in translating the structuring element at every position in space and check if this translated structuring element is included in (respectively intersects) the original set [70] (see Section 2). This principle has also been used in the main extensions of mathematical morphology to fuzzy sets, in the form of degree of inclusion and degree of intersection as explained in Section 3. Similarly, defining morphological erosions and dilations of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion and a degree of intersection between bipolar fuzzy sets. Such degrees have been proposed for instance in the context of intuitionistic fuzzy sets [5,39]. Although their semantics are different from the ones of bipolar fuzzy sets [42], formally we can use similar ideas.

4.3.1. Bipolar connectives

Degrees of intersection and inclusion rely on connectives. Fuzzy connectives extend to bipolar ones as follows:

- A bipolar negation, or complementation, on $\mathcal{L}$ is a decreasing operator N such that $N((0,1)) = (1,0)$ and $N((1,0)) = (0,1)$. In this paper, we restrict ourselves to involutive negations, such that $\forall(a,b) \in \mathcal{L},\ N(N((a,b))) = (a,b)$. An example, which will be used in the following, is the standard negation, defined by $N((a,b)) = (b,a)$.
- A bipolar conjunction is an operator C from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $C((0,1),(0,1)) = C((0,1),(1,0)) = C((1,0),(0,1)) = (0,1),\ C((1,0),(1,0)) = (1,0)$, and that is increasing in both arguments.
- A bipolar t-norm is a commutative and associative bipolar conjunction such that $\forall(a,b) \in \mathcal{L},\ C((a,b),(1,0)) = (a,b)$ (i.e. the largest element of $\mathcal{L}$ is the unit element of C). It follows that the smallest element is the null element: $\forall(a,b) \in \mathcal{L},\ C((a,b),(0,1)) = (0,1)$.
- A bipolar disjunction is an operator D from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $D((1,0),(1,0)) = D((0,1),(1,0)) = D((1,0),(0,1)) = (1,0),\ D((0,1),(0,1)) = (0,1)$, and that is increasing in both arguments.
- A bipolar t-conorm is a commutative and associative bipolar disjunction such that $\forall(a,b) \in \mathcal{L}, D((a,b),(0,1)) = (a,b)$ (i.e. the smallest element of $\mathcal{L}$ is the unit element of D). It follows that the largest element is the null element: $\forall(a,b) \in \mathcal{L},\ D((a,b),(1,0)) = (1,0)$.
- A bipolar implication is an operator I from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $I((0,1),(0,1)) = I((0,1),(1,0)) = I((1,0),(1,0)) = (1,0),\ I((1,0),(0,1)) = (0,1)$, and that is decreasing in the first argument and increasing in the second argument.

Two types of conjunctions and disjunctions are considered in [39] and will be considered here as well:

1. operators called t-representable t-norms and t-conorms, which can be expressed using usual t-norms T and t-conorms T^C (see Section 3.1):

$$C((a_1,b_1),(a_2,b_2)) = \left(T(a_1,a_2), T^C(b_1,b_2)\right), \tag{13}$$

$$D((a_1,b_1),(a_2,b_2)) = \left(T^C(a_1,a_2), T(b_1,b_2)\right). \tag{14}$$

A typical example is obtained for $T = \min$ and $T^C = \max$.

2. Lukasiewicz operators, which are not t-representable:

$$C_W((a_1,b_1),(a_2,b_2)) = (\max(0, a_1 + a_2 - 1), \min(1, b_1 + 1 - a_2, b_2 + 1 - a_1)), \tag{15}$$

$$D_W((a_1,b_1),(a_2,b_2)) = (\min(1, a_1 + 1 - b_2, a_2 + 1 - b_1), \max(0, b_1 + b_2 - 1)). \tag{16}$$

In these equations, the positive part of C_W is the usual Lukasiewicz t-norm of a_1 and a_2 (i.e. the positive parts of the input bipolar values). The negative part of D_W is the usual Lukasiewicz t-norm of the negative parts (b_1 and b_2) of the input values.

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As in the fuzzy case (see Section 3.1), two types of implications can be defined [39,33], one derived from a bipolar t-conorm D and a negation N:

$$I_N((a_1,b_1),(a_2,b_2)) = D(N(a_1,b_1),(a_2,b_2)), \tag{17}$$

and one derived from a residuation principle from a bipolar t-norm C:

$$I_R((a_1,b_1),(a_2,b_2)) = \sup\{(a_3,b_3) \in \mathcal{L} \mid C((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2)\}, \tag{18}$$

where (a_i,b_i) are elements of $\mathcal{L}$. The two types of implication coincide for the Lukasiewicz operators [34].

Here are some other ways to link these connectives to each other:

- a t-conorm D can be derived from a t-norm C and a negation N as: $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$
  $D((a_1,b_1),(a_2,b_2)) = N(C(N((a_1,b_1)),N((a_2,b_2))))$;
- an implication I induces a negation N defined as:
  $\forall(a,b) \in \mathcal{L}, N((a,b)) = I((a,b),(0,1))$;
- an implication I can be defined from a negation N and a conjunction C by: $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$
  $I((a_1,b_1),(a_2,b_2)) = N(C((a_1,b_1),N((a_2,b_2))))$;
- conversely, a conjunction C can be defined from a negation N and an implication I: $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$
  $C((a_1,b_1),(a_2,b_2)) = N(I((a_1,b_1),N((a_2,b_2))))$.

4.3.2. Bipolar erosions

Using these notations, a degree of inclusion of a bipolar fuzzy set (μ',ν') in another bipolar fuzzy set (μ,ν) is defined as:

$$\bigwedge_{x\in\mathcal{S}} I((\mu'(x),\nu'(x)),(\mu(x),\nu(x))), \tag{19}$$

where I is an implication operator.

A similar approach has been used for intuitionistic fuzzy sets in [60], but with weaker properties (in particular an important property such as the commutativity of erosion with the conjunction may be lost).

Based on these concepts, we can now propose a definition for morphological erosion.

Definition 5. Let (μ_B,ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The erosion of any (μ,ν) in $\mathcal{B}$ by (μ_B,ν_B) is defined from an implication I as:

$$\forall x \in \mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigwedge_{y\in\mathcal{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y))). \tag{20}$$

In this equation, $\mu_B(y-x)$ (respectively $\nu_B(y-x)$) represents the value at point y of the translation of μ_B (respectively ν_B) at point x.

In order to interpret this definition, let us consider the implication defined from a t-representable bipolar t-conorm D and the standard negation. Then the erosion can be rewritten as:

$$\begin{aligned}\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \bigwedge_{y\in\mathcal{S}} D((\nu_B(y-x),\mu_B(y-x)),(\mu(y),\nu(y))) = \bigwedge_{y\in\mathcal{S}} (T^C(\nu_B(y-x),\mu(y)),T(\mu_B(y-x),\nu(y)))\\ &= \left(\inf_{y\in\mathcal{S}} T^C(\nu_B(y-x),\mu(y)), \sup_{y\in\mathcal{S}} T(\mu_B(y-x),\nu(y))\right).\end{aligned} \tag{21}$$

The second line is derived from the fact that D is supposed here to be a t-representable bipolar t-conorm, defined from a t-norm T and a t-conorm T^C. The third line is derived from the definition of the infimum in $\mathcal{L}$ and in $\mathcal{B}$. This resulting bipolar fuzzy set has a membership function (positive part) which is exactly the fuzzy erosion of μ by the fuzzy structuring element $1-\nu_B$, according to the original definitions in the fuzzy case [24] (see Section 3.3). The non-membership function (negative part) is exactly the dilation of the fuzzy set ν by the fuzzy structuring element μ_B.

4.3.3. Bipolar dilations

Dilation can be defined based on a duality principle or based on the adjunction property. Both approaches have been developed in the case of fuzzy sets, and the links between them and the conditions for their equivalence have been proved in [13,19]. Similarly we consider both approaches to define morphological dilation on $\mathcal{B}$.

The duality principle states that the dilation is equal to the complementation of the erosion, by the same structuring element (if it is symmetrical with respect to the origin of $\mathcal{S}$, otherwise its symmetrical with respect to the origin of space $\mathcal{S}$ is used), applied to the complementation of the original set. Applying this principle to bipolar fuzzy sets using a complementation N (typically the standard negation $N((a,b)) = (b,a)$) leads to the following definition of morphological bipolar dilation.

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Definition 6. Let (μ_B, ν_B) be a bipolar fuzzy structuring element. The dilation of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from erosion by duality as:

$$\delta_{(\mu_B,\nu_B)}((\mu, \nu)) = N[\varepsilon_{(\mu_B,\nu_B)}(N((\mu, \nu)))]. \tag{22}$$

Let us now consider the adjunction principle, as in the general algebraic case. An adjunction property can also be expressed between a bipolar t-norm C and the corresponding residual implication I_R (see Eq. (18)) as follows:

$$C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2) \iff (a_3, b_3) \preceq I_R((a_1, b_1), (a_2, b_2)). \tag{23}$$

Definition 7. Using a residual implication for the erosion for a bipolar t-norm C, the bipolar fuzzy dilation, adjoint of the erosion, is defined as:

$$\delta_{(\mu_B,\nu_B)}((\mu, \nu)) = \bigwedge \{(\mu', \nu') \in \mathcal{B} \,|\, (\mu, \nu) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu', \nu'))\}, \tag{24}$$

i.e.

$$\forall x \in \mathcal{S}, \quad \delta_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigvee\nolimits_{y\in\mathcal{S}} C((\mu_B(x-y), \nu_B(x-y)), (\mu(y), \nu(y))). \tag{25}$$

It is easy to show that the bipolar Lukasiewicz operators are adjoint, according to Eq. (23). It has been shown that the adjoint operators are all derived from the Lukasiewicz operators, using a continuous bijective permutation on $[0,1]$ [39]. They are also dual with respect to the standard negation. Hence equivalence between both approaches can be achieved only for this class of operators.

4.3.4. Properties

Properties of these operations are consistent with the ones holding for sets and for fuzzy sets, and are detailed in [15,16,18,20,17] (they are summarized in the next proposition). Interpretations of these definitions as well as some illustrative examples can also be found in these references.

Proposition 2. *The following properties hold:*

- *All definitions are consistent: they actually provide bipolar fuzzy sets of $\mathcal{B}$, i.e. $\forall(\mu, \nu) \in \mathcal{B},\ \forall(\mu_B, \nu_B) \in \mathcal{B},\ \delta_{(\mu_B,\nu_B)}((\mu, \nu)) \in \mathcal{B}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \in \mathcal{B}$.*
- *In case the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1 - \mu$ and $\nu_B = 1 - \mu_B$), the definitions lead to the usual definitions of fuzzy dilations and erosions (Definition 2). Hence they are also compatible with classical morphology in case μ and μ_B are crisp.*
- *If the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1 - \mu$ and $\nu_B = 1 - \mu_B$), the definitions based on the bipolar Lukasiewicz operators are equivalent to Definition 2 obtained for the classical Lukasiewicz t-norm and t-conorm.*
- *The proposed definitions of bipolar fuzzy dilations and erosions commute respectively with the supremum and the infimum of the lattice $(\mathcal{B}, \preceq)$.*[2]
- *If Lukasiewicz operators (up to a bijection) are used, then all algebraic properties detailed in Section 2 hold.*[3]
- *The bipolar fuzzy dilation is extensive (i.e. $(\mu, \nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu, \nu))$) and the bipolar fuzzy erosion is anti-extensive (i.e. $\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \preceq (\mu, \nu)$) if and only if $(\mu_B, \nu_B)(0) = (1, 0)$, where 0 is the origin of the space $\mathcal{S}$ (i.e. the origin completely belongs to the structuring element, without any indetermination).*[4]
- *The following iterativity (i.e. associativity) property holds*[5]*:*

$$\delta_{(\mu_B,\nu_B)}\left(\delta_{(\mu'_B,\nu'_B)}((\mu, \nu))\right) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu, \nu)). \tag{26}$$

- *The conjunctions and disjunctions involved in the definitions have actually to be bipolar t-norms and t-conorms in order to have all classical morphological properties. More general conjunctions and disjunctions are not interesting from a mathematical morphology point of view, since they are rather weak from the point of view of their properties.*[6]

4.4. A note on semantics and partial ordering

It is interesting to note that bipolar fuzzy sets are formally linked to intuitionistic fuzzy sets [5], interval-valued fuzzy sets [74] and vague sets, or to clouds when boundary constraints are added [62,40], as shown by several authors [42,30].

[2] This property is obvious in the case of adjunctions. In the more general case, for definitions derived from any bipolar t-norm C and bipolar t-conorm D, it results from the fact that C distributes over the supremum and D over the infimum.

[3] Since in this case the adjunction property holds.

[4] This is derived from the fact that $(1,0)$ is the unit element of C, and $(0,1)$ is the unit element of D. Note that this condition is equivalent to the conditions on the structuring element found in classical and fuzzy morphology to have extensive dilations and anti-extensive erosions [70,24].

[5] It is derived from the commutativity and associativity of bipolar t-norms.

[6] For instance, the iterativity property is lost if the conjunction is not associative.

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However, their respective semantics differ. Concerning intuitionistic fuzzy sets, there has been a lot of discussion about terminology in this domain recently [42,6], mainly because of the word "intuitionistic" which is misleading and introduces confusions with intuitionistic logics. The semantics of intuitionistic fuzzy sets reveals the possibility of some indetermination between membership and non-membership to a set. As for interval-valued fuzzy sets or clouds, their semantics correspond to the representation of some imprecision or uncertainty about the membership value, which can only be given as an interval, and not as a crisp number.

However the semantics of bipolar fuzzy sets is different. A bipolar fuzzy set in the spatial domain does not necessarily represent one physical object or spatial entity, but rather more complex information, potentially issued from different sources. This refers to the third type of bipolarity, according to the classification presented in [43,46]. What we call a bipolar fuzzy set is then a mathematical complex object, not a physical object. An example is the modeling of information concerning the location of a robot: positive information could concern potential locations (for instance derived from sensor data) and negative information could concern forbidden places (because they are already occupied by other objects, or the robot is not allowed to move there, etc.). Let us consider another example, in a different domain, about preference modeling [8]. Positive information describes what is desired and allows sorting solutions, while negative information describes what is rejected or unacceptable and defines constraints to be fulfilled. The gap between positive and negative information does not necessarily concern indetermination, but rather neutrality or indifference.

In this section, we developed a theory for mathematical morphology on bipolar fuzzy sets based on Pareto partial ordering. This implies that positive information and negative information play symmetrical roles. However, based on the above discussion about semantics, this might not always be appropriate, since we may want to process positive and negative information in different ways. In the particular context of mathematical morphology, this ordering has an additional drawback: the value at a point in the resulting dilation or erosion is generally expected to be one of the values of neighborhood points (defined by the structuring element), but this is in general not the case in the proposed approach. This point has already raised discussions in the mathematical morphology community, in particular when dealing with vector-valued images, such as color images (see e.g. [2,4,73]). It has been shown that non vector-preserving orderings may lead to counter-intuitive results (for instance introducing new colors, that do not belong to any of the image objects, may prevent their correct recognition).

In the next section, we therefore introduce priorities between the two types of information, based on a lexicographic ordering which induces another way of modeling mathematical morphology, and which guarantees that the resulting bipolar value at a point is one of the values of neighborhood points.

5. Bipolar fuzzy mathematical morphology based on lexicographic ordering

As mentioned before, the partial ordering $\preceq$ exploited in Section 4 processes positive and negative information in a similar way. This is not always convenient when the two types of information are issued from different sources or have different semantics. For instance if the positive information represents preferences and the negative information rules or constraints, then it may be interesting (or mandatory) to give more priority to the constraints (or in the contrary to the positive information) [8,9,47,52,55,64]. The partial ordering $\preceq$ should then be replaced by another one, accounting for these priorities. Here we consider the lexicographic ordering (also called dictionary ordering), denoted by $\preceq_L$, which matches these requirements, and which is additionally a total order on $\mathcal{L}$. On the induced lattice on $\mathcal{B}$, we define algebraic dilations and erosions. We also propose connectives that are adapted to this ordering, and then derive morphological dilations and erosions. This extends a preliminary work in [21].

5.1. Lexicographic ordering and associated lattice

Definition 8. The lexicographic relation $\preceq_L$ on $\mathcal{L}$, giving priority to negative information, is defined as:

$$(a,b)\preceq_L(a',b') \iff b > b' \quad \text{or} \quad (b = b' \text{ and } a \leqslant a'). \tag{27}$$

Proposition 3. *The relation $\preceq_L$ defines a total ordering on $\mathcal{L}$ and $(\mathcal{L},\preceq_L)$ is a complete lattice. The smallest element is $(0,1)$ and the largest element is $(1,0)$.*

A lexicographic ordering giving priority to the positive information can be defined in a similar way. All what follows applies in both cases, and we only detail the one of Definition 8 in this paper.

Fig. 1 illustrates the difference between $\preceq$ and $\preceq_L$.

This ordering induces a partial ordering on $\mathcal{B}$ (the same notation is used):

Definition 9. The lexicographic relation on $\mathcal{B}$ is defined by:

$$(\mu,\nu)\preceq_L(\mu',\nu') \iff \forall x \in \mathcal{S},\ (\mu(x),\nu(x))\preceq_L(\mu'(x),\nu'(x)). \tag{28}$$

This definition means that a bipolar fuzzy set is considered as smaller than another one if its negative part is larger, or if the two negative parts are equal and the positive part is smaller. This strongly expresses the priority given to the negative information, since only the negative parts are considered as soon as they differ.

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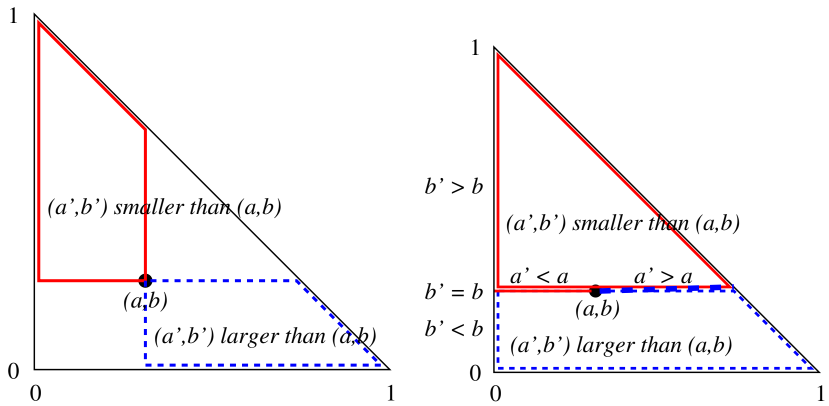

Fig. 1. Comparison, in $\mathcal{L}$, between the partial ordering $\preceq$ (left) and the total ordering $\preceq_L$ (right). Plain (respectively dashed) lines indicate the regions of $\mathcal{L}$ in which points (a', b') are smaller (respectively larger) than point (a, b).

Proposition 4. *The relation $\preceq_L$ (Definition 9) defines a partial ordering, called lexicographic ordering, on $\mathcal{B}$ and $(\mathcal{B}, \preceq_L)$ is a complete lattice. The smallest element is (μ_0, ν_0) (defined by $\forall x \in \mathcal{S},\ \mu_0(x) = 0,\ \nu_0(x) = 1$), and the largest element is $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ (defined by $\forall x \in \mathcal{S},\ \mu_{\mathbb{I}}(x) = 1,\ \nu_{\mathbb{I}}(x) = 0$).*

Proposition 5. *Infimum and supremum for $\preceq_L$ are expressed, for any two elements (a,b) and (a', b') of $\mathcal{L}$, as:*

$$\min_{\preceq_L}((a,b),(a',b')) = \begin{cases} (a,b) & \text{if } b > b', \\ (a',b') & \text{if } b < b', \\ (\min(a,a'),b) & \text{if } b = b', \end{cases} \tag{29}$$

$$\max_{\preceq_L}((a,b),(a',b')) = \begin{cases} (a,b) & \text{if } b < b', \\ (a',b') & \text{if } b > b', \\ (\max(a,a'),b) & \text{if } b = b'. \end{cases} \tag{30}$$

Infimum and supremum for any family of elements of $\mathcal{L}$ or $\mathcal{B}$ are derived in a straightforward way, and are denoted by $\bigwedge_{\preceq_L}$ and $\bigvee_{\preceq_L}$.

Let us note that, in all cases, the lexicographic minimum (or maximum) provides a result which is one of the input bipolar values, and the following equivalences hold:

$$\min_{\preceq_L}((a,b),(a',b')) = (a,b) \iff (a,b) \preceq_L (a',b'),$$
$$\max_{\preceq_L}((a,b),(a',b')) = (a,b) \iff (a,b) \succeq_L (a',b').$$

5.2. Connectives

Bipolar connectives are defined as in Section 4. However, it should be noticed that, in these definitions, the notion of monotonicity depends on the considered ordering defined on $\mathcal{L}$. Here we then have to consider monotonicity with respect to $\preceq_L$.

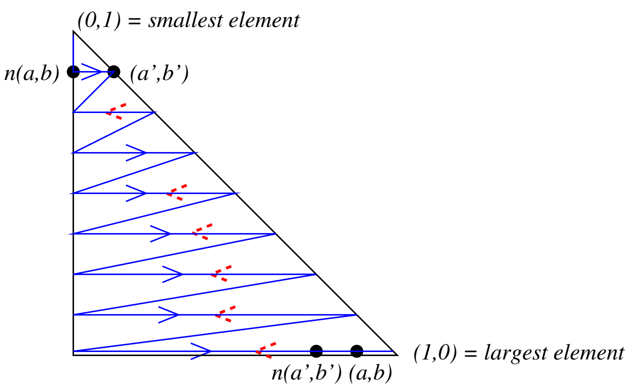

Fig. 2. Natural negation for the lexicographic ordering. Plain arrows indicate the ordering from the smallest to the largest element of $\mathcal{L}$ and the dashed arrows indicate the reverse order. Two examples of points (a, b) and (a', b') and their negations $n_{\preceq_L}(a, b)$ and $n_{\preceq_L}(a', b')$ are shown.

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With respect to the Pareto ordering $\preceq$, the standard negation $N((a,b)) = (b,a)$ is decreasing. However it is not for the lexicographic ordering $\preceq_L$ and is hence not a negation. Therefore, we propose a new definition of negation, illustrated in Fig. 2.

Definition 10. The natural negation $n_{\preceq_L}$ associated with the lexicographic ordering is defined as the operator that reverses the ordering of the elements of $\mathcal{L}$.

This definition of $n_{\preceq_L}$ is actually a negation (involutive and decreasing). This result is derived from the fact that $\preceq_L$ is a total ordering on $\mathcal{L}$.

From an algorithmical point of view, the computation of the negation is simple when the levels between 0 and 1 are discrete, i.e. take only a finite number of values (which is generally the case in practical applications). We tabulate the ranks of (a_i, b_j), for i and j varying from 0 to N if the interval $[0,1]$ is discretized on $N+1$ levels (for instance $a_i = \frac{i}{N},\ b_j = \frac{j}{N}$). The rank of $(\frac{i}{N}, \frac{j}{N})$ is $r_{ij} = \frac{(N-j+1)(N-j)}{2} + i$ and the rank of $n_{\preceq_L}(\frac{i}{N}, \frac{j}{N})$ is equal to $\frac{(N+1)(N+2)}{2} - 1 - r_{ij}$.

From a geometrical point of view, the negation of a point (a,b) is the point $n_{\preceq_L}(a,b)$ such that the number of points in the triangle comprising the points smaller than (a,b) (see Fig. 1) is equal to the number of points in the trapeze formed by the points that are larger than $n_{\preceq_L}(a,b)$.

Proposition 6. *The minimum* $\min_{\preceq_L}$ *and maximum* $\max_{\preceq_L}$ *associated with the lexicographic ordering are bipolar t-norms and t-conorms on the lattice* $(\mathcal{L}, \preceq_L)$.[7] *Moreover they are idempotent and mutually distributive,* $\min_{\preceq_L}$ *is the largest t-norm and* $\max_{\preceq_L}$ *the smallest t-conorm (according to* $\preceq_L$*). They are also dual with respect to the negation* $n_{\preceq_L}$.

Proposition 7. *The connective* I_M *defined as* $\forall (a,b) \in \mathcal{L},\ \forall (a',b') \in \mathcal{L}$

$$I_M((a,b),(a',b')) = \max_{\preceq_L} \left(n_{\preceq_L}(a,b),(a',b') \right), \tag{31}$$

is a bipolar implication.

Conversely, the negation can be deduced from the implication according to: $n_{\preceq_L}(a,b) = I_M((a,b),(0,1))$.

Proposition 8. *The connective* I_R *defined as* $\forall (a,b) \in \mathcal{L},\ \forall (a',b') \in \mathcal{L}$

$$I_R((a,b),(a',b')) = \bigvee\nolimits_{\preceq_L} \left\{ (\alpha,\beta) \in \mathcal{L} \mid \min_{\preceq_L}((a,b),(\alpha,\beta)) \preceq_L (a',b') \right\}, \tag{32}$$

is a bipolar implication (it is the residual implication of the t-norm $\min_{\preceq_L}$*). A close form expression is as follows:*

$$I_R((a,b),(a',b')) = \begin{cases} (1,0) & \text{if } b > b', \\ (a',b') & \text{if } b < b', \\ (1,0) & \text{if } b = b' \text{ and } a \leqslant a', \\ (a',b') & \text{if } b = b' \text{ and } a > a', \end{cases} \tag{33}$$

or, equivalently:

$$I_R((a,b),(a',b')) = \begin{cases} (1,0) & \text{if } (a,b) \preceq_L (a',b'), \\ (a',b') & \text{if } (a',b') \prec_L (a,b). \end{cases} \tag{34}$$

It is the adjoint of $\min_{\preceq_L}$*, i.e.:* $\forall (a_i,b_i) \in \mathcal{L},\ i = 1,\ldots,3$

$$\min_{\preceq_L}((a_1,b_1),(a_2,b_2)) \preceq_L (a_3,b_3) \iff (a_2,b_2) \preceq_L I_R((a_1,b_1),(a_3,b_3)). \tag{35}$$

This result is important for the construction of morphological operations, as will be seen next.

5.3. Algebraic dilations and erosions on the lattice $(\mathcal{B}, \preceq_L)$

Since $(\mathcal{B}, \preceq_L)$ is a complete lattice, algebraic dilations and erosions can be defined as in Section 2.

Definition 11. A dilation is an operator δ from $\mathcal{B}$ into $\mathcal{B}$ which commutes with the supremum:

$$\forall (\mu_i, \nu_i) \in \mathcal{B}, \quad \delta\left(\bigvee\nolimits_{\preceq_L} (\mu_i, \nu_i)\right) = \bigvee\nolimits_{\preceq_L} \delta((\mu_i, \nu_i)). \tag{36}$$

An erosion is an operator ε from $\mathcal{B}$ into $\mathcal{B}$ which commutes with the infimum:

$$\forall (\mu_i, \nu_i) \in \mathcal{B}, \quad \varepsilon\left(\bigwedge\nolimits_{\preceq_L} (\mu_i, \nu_i)\right) = \bigwedge\nolimits_{\preceq_L} \varepsilon((\mu_i, \nu_i)). \tag{37}$$

[7] Note that $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are not increasing with respect to $\preceq$ and are therefore not *t*-norms and *t*-conorms on $(\mathcal{L}, \preceq)$.

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Definition 12. A pair of operators (ε, δ) is an adjunction on $(\mathcal{B}, \preceq_L)$ if:

$$\forall(\mu, \nu) \in \mathcal{B},\ \forall(\mu', \nu') \in \mathcal{B}, \quad \delta((\mu, \nu)) \preceq_L (\mu', \nu') \iff (\mu, \nu) \preceq_L \varepsilon((\mu', \nu')). \tag{38}$$

The properties of these operators and their compositions (in particular closing and opening) are directly derived from the properties of complete lattices and are the same as those described in Section 2.

5.4. Morphological dilations and erosions on the lattice $(\mathcal{B}, \preceq_L)$

Let us now again consider the case where $\mathcal{S}$ is an affine space, on which translations are defined. Again we define a degree of intersection as the supremum of a bipolar conjunction and a degree of inclusion as the infimum of a bipolar implication, according to $\preceq_L$.

Definition 13. Let (μ_B, ν_B) be a bipolar structuring element (in $\mathcal{B}$). The dilation of any element (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar t-norm C as:

$$\forall x \in \mathcal{S}, \quad \delta_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigvee\nolimits_{\preceq_L y \in \mathcal{S}} C((\mu_B(x - y), \nu_B(x - y)), (\mu(y), \nu(y))). \tag{39}$$

The erosion of (μ, ν) by (μ_B, ν_B) is defined from a bipolar implication I as:

$$\forall x \in \mathcal{S}, \quad \varepsilon_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigwedge\nolimits_{\preceq_L y \in \mathcal{S}} I((\mu_B(y - x), \nu_B(y - x)), (\mu(y), \nu(y))). \tag{40}$$

In particular, we can use the lexicographic minimum $\min_{\preceq_L}$ as a t-norm. An example is illustrated in Fig. 3.

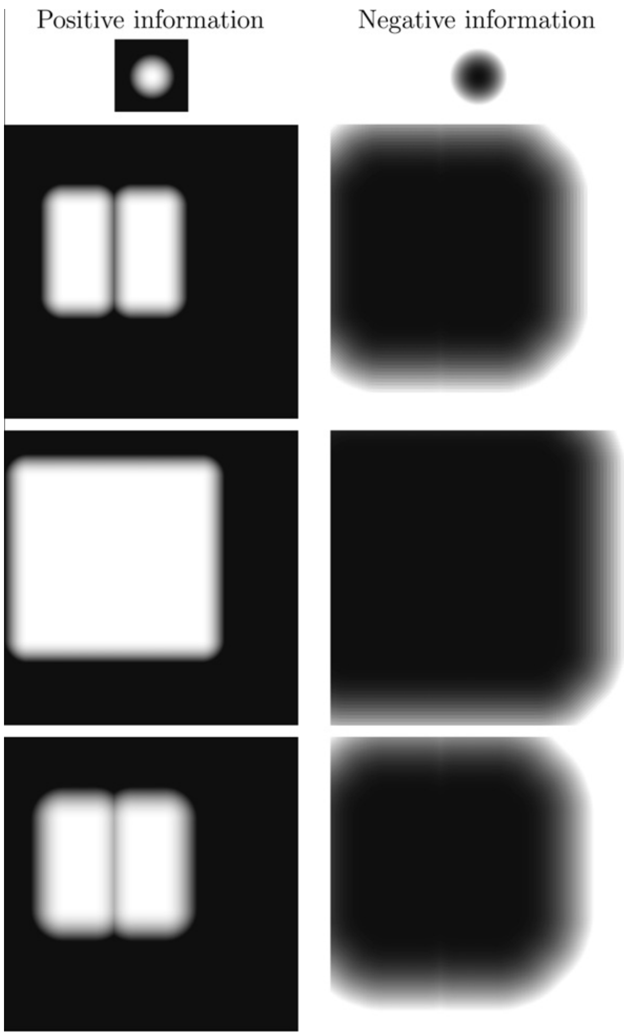

Fig. 3. From top to bottom: bipolar fuzzy structuring element, original bipolar fuzzy set, dilation using the lexicographic minimum, dilation using Pareto ordering, for the sake of comparison. The grey levels encode the membership (or non-membership) values, ranking from 0 (black) to 1 (white).

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As expected, the dilation extends the positive parts and reduces the negative parts. The priority given to the negative parts and the fact that $\min_{\preceq_L}$ always provides one of the input values (which is not the case of the Pareto ordering) induces a stronger effect of the transformation when using the lexicographic ordering (the Pareto minimum has the same negative part than $\min_{\preceq_L}$ and a smaller positive part).

It should be noted that, as in Section 4, a t-norm is involved in the proposed definition (i.e. a stronger operator than a general bipolar conjunction), so as to guarantee good properties. For the erosion, both types of implications I_M and I_R can be used, with somewhat different properties.

Proposition 9. *The dilation defined from* $\min_{\preceq_L}$ *and the erosion defined from* I_M *(for* $\max_{\preceq_L}$ *and the negation* $n_{\preceq_L}$*) are dual with respect to the negation* $n_{\preceq_L}$ *:* $\delta_{(\mu_B,\nu_B)}(n_{\preceq_L}(\mu,\nu)) = n_{\preceq_L}(\varepsilon_{(\mu_B,\nu_B)}(\mu,\nu))$.

Proposition 10. *The dilation defined from* $\min_{\preceq_L}$ *and the erosion defined from* I_R *(residual implication of* $\min_{\preceq_L}$*) are adjoint. It follows that all general algebraic properties described in Section 2 hold.*

Note that the two properties of adjunction and of duality are not simultaneously satisfied for these operators (since the dual operator of $\min_{\preceq_L}$ is $\max_{\preceq_L}$ but it is not its adjoint). It would be interesting to prove the existence and then build operators equivalent to Lukasiewicz ones, for $\preceq_L$, so as to derive similar results as in the fuzzy case and in the bipolar fuzzy case for the Pareto ordering (see Sections 3 and 4).

It follows that the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are true opening and closing if $\min_{\preceq_L}$ and I_R are used (because of the adjunction property), while they are not if $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are used (they are not idempotent in this case).

Proposition 11. *Dilation and erosion defined by Eqs. (39) and (40) form an adjunction if and only if the involved C and I operators are adjoint. The general algebraic properties then hold (see Section 2).*

Proposition 12. *The following properties hold:*

- *Definition 13 is consistent and provides results in* $\mathcal{B}$.
- *The dilation commutes with the supremum and the erosion with the infimum of the lattice* $(\mathcal{B},\preceq_L)$.[8]
- *Both operations are increasing with respect to* $\preceq_L$.[9]
- *The dilation is extensive and the erosion is anti-extensive if and only if the origin of* $\mathcal{B}$ *completely belongs to the structuring element (i.e.* $(\mu_B,\nu_B)(0) = (1,0)$*).*[10]
- *In the particular case where the set and the structuring element are not bipolar (*$\nu = 1-\mu$ *and* $\nu_B = 1-\mu_B$*), the definitions reduce to the classical ones in the fuzzy case.*
- *The following iterativity property holds*[11]*:*

$$\delta_{(\mu_B,\nu_B)}\left(\delta_{(\mu'_B,\nu'_B)}((\mu,\nu))\right) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu,\nu)). \tag{41}$$

6. Discussion and related works

One of the main issues in the proposed extensions of mathematical morphology to bipolar fuzzy sets is to handle properly the two components (i.e. positive and negative information). We proposed two different ways in this paper, based on Pareto and lexicographic ordering, respectively. Although both lead to the adequate framework of complete lattices, they have some drawbacks: the Pareto ordering handles both components in a symmetric way, which may not be suitable according to the semantics of the bipolarity; the lexicographic ordering on the contrary gives a strong priority to one component, and the other one is then seldom considered. This might be to strict for some problems, but for some others, for instance when the negative information represents constraints that cannot be overcome, it may be interesting. Such problems have been addressed in other types of work, where different partial orderings have been discussed. We mention here two examples: color image processing and social choice.

Color image processing: The question of defining a suitable ordering on vectorial images (in particular color images) has been widely addressed in the mathematical imaging community (see e.g. [3] for a review). The lexicographic ordering is known to excessively privilege one of the colors, and can be refined for instance by defining a rougher quantization on the first color, so as to have more frequent comparisons based on the two other ones. Other approaches use a 1D scale by

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[8] This property is obvious in the case of adjunctions. In the more general case, it results from the fact that C distributes over the supremum and D over the infimum.

[9] Again this is straightforward in case of adjunctions. In the more general case, this is derived from the increasingness of the bipolar connectives and of the infimum and supremum with respect to $\preceq_L$.

[10] This comes from the fact that $(1,0)$ is the unit element of the bipolar t-norms, and $(0,1)$ of the t-conorms.

[11] directly derived from the associativity of the t-norm.

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applying a scalar function to the vectors, define groups of vectors which are then ranked, or base the comparison on a distance to a reference vector, just to cite a few ones [3,4,2].

Social choice: This is another domain where the question of defining a partial ordering is crucial, in particular for multi-criteria decision making or voting problems. Various orderings have been proposed, including refinements of the lexicographic ordering, leximin/leximax, discrimax, tolerant Pareto, etc. [58,69,29,41,48].

It would be interesting to continue our work on bipolar mathematical morphology using one of these orderings and to study their properties and their adequation depending on the domain of application and on the associated semantics.

7. Conclusion

The framework of complete lattices is the basis of mathematical morphology operators. In this paper we have developed two extensions of mathematical morphology, for the lattice of fuzzy sets on the one hand, and for the lattice of bipolar fuzzy sets on the other hand, using two different partial orderings. Thanks to the strong algebraic structure provided by the lattice framework, these extensions inherit all classical properties of mathematical morphology.

We focused here on the mathematical constructions of the basic operators. It is clear that combined operators can be derived [16,20], as well as several applications, in particular for spatial reasoning, structural object recognition in images and logical reasoning [14,32,61,51,23,27]. First applications of bipolar fuzzy sets for spatial reasoning in images can be found in [16,17].

Future work aims at investigating other orderings, and at developing the applications in particular for the bipolar case, in the domain of spatial reasoning and preference representations. Extensions to semi-lattices or general posets could be interestingly considered as well.

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