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Mathematical morphology on bipolar fuzzy sets: General algebraic framework

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ABSTRACT

In many domains of information processing, bipolarity is a core feature to be considered: positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. If the information is moreover endowed with vagueness and imprecision, as is the case for instance in spatial information processing, then bipolar fuzzy sets constitute an appropriate knowledge representation framework. In this paper, we focus on mathematical morphology as a tool to handle such information and reason on it. Applying mathematical morphology to bipolar fuzzy sets requires defining an appropriate lattice. We extend previous work based on specific partial orderings to any partial ordering leading to a complete lattice. We address the case of algebraic operations and of operations based on a structuring element, and show that they have good properties for any partial ordering, and that they can be useful for processing in particular spatial information, but also other types of bipolar information such as preferences and constraints. Particular cases using Pareto and lexicographic orderings are illustrated. Operations derived from fuzzy bipolar erosion and dilation are proposed as well.

1. Introduction

A recent trend in contemporary information processing focuses on bipolar information, both from a knowledge representation point of view, and from a processing and reasoning one. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [47,49]. This domain has recently motivated work in several directions, for instance for applications in knowledge representation, preference modeling, argumentation, multi-criteria decision analysis, cooperative games, among others [2,10,23,30,49,56,65,67,81,82,85]. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [9,10,47,50]. Three types of bipolarity are distinguished in [50]: (i) symmetric univariate, where a unique totally ordered scale covers the range from negative (not satisfactory) to positive (satisfactory) information (e.g., modeled by probabilities); (ii) symmetric bivariate, where two separate scales are linked together and concern related information (e.g., modeled by belief functions); (iii) asymmetric or heterogeneous, where two types of information are not necessarily linked together and may come from different sources. This last type is particularly interesting in image interpretation and spatial reasoning.

In this paper, we propose to handle such bipolar information using mathematical morphology operators. Mathematical morphology [91,92,95] has proved to be useful to process information in many different domains, such as image and vision, spatial reasoning, preference modeling and logics (for fusion, revision, abduction, mediation, . . .). Extending mathematical morphology to bipolar information will therefore increase the modeling and reasoning capabilities in all these domains.

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This extension can be performed in a generic way, by defining a lattice as the underlying structure of bipolar knowledge representation (the interest of using complete lattices for mathematical morphology has been justified in [86]). The general framework of mathematical morphology leads to the definition of algebraic dilations and erosions, which are the two main operators, from which other ones can then be derived. This general formalism applies in different settings, and the proposed definitions can be specified for different types of lattices, e.g., based on bipolar sets, fuzzy sets or logical formulas. Bipolar fuzzy sets can be seen as a structure covering several settings, and is therefore considered in this paper. Moreover, it allows handling an additional feature of imperfect information, related to its imprecision. Hence the proposed framework allows representing and dealing with both bipolarity and fuzziness. The main contribution of this paper is to show that previous work using specific partial orderings extends to any partial ordering endowing bipolar fuzzy sets with a complete lattice structure. This flexibility in the choice of the partial ordering allows the same framework to be adapted, among others, to different types of bipolarity. Moreover, derived operators are suggested, and a preliminary discussion is proposed about the type of bipolarity and semantics, in particular for spatial information processing applications, and the choice of partial ordering.

Mathematical morphology on bipolar fuzzy sets was proposed for the first time in [15], by considering the complete lattice defined from the Pareto ordering. Then it was further developed, with additional properties, geometric aspects and applications to spatial reasoning, in [16,18]. The lexicographic ordering was considered too in [19]. Here we go one step further by considering any partial ordering, and also proposing derived operators. Similar work has been developed independently, in the setting of intuitionistic fuzzy sets and interval-valued fuzzy sets, also based on Pareto ordering [73,78,79]. This group also addressed links with the binary case, for construction and decomposition of operators based on $[\alpha_1, \alpha_2]$-cuts [74, 75], and proposed recently an extension to L-fuzzy sets [97], besides their important contribution to connectives (e.g., [34,40–42]). Here, while relying on the general algebraic framework of mathematical morphology on the one hand, and on L-fuzzy sets [55] on the other hand, we restrict ourselves to the special case of bipolar fuzzy sets, according to Definition 1 below, but considering any partial ordering as a new contribution.

In Section 2, we set the algebraic framework, by defining a lattice structure on bipolar information, and introducing connectives. The remaining of the paper will rely on a representation of bipolar information as bipolar fuzzy sets, which encompasses several other models. Definitions of algebraic dilations and erosions of bipolar fuzzy sets are given in Section 3, in a general way, whatever the chosen partial ordering. In the spatial domain, specific forms of these operators, involving a structuring element, are particularly interesting [91]. They are called morphological dilations and erosions. More generally they are useful in any application where some relation between elements of the underlying space should be involved. Morphological erosions and dilations are then defined, and their properties are discussed, still for any partial ordering. In the next two sections, we detail the case of two particular partial orderings: Pareto (or marginal) ordering in Section 4 and lexicographic ordering in Section 5. Finally, some derived operators are introduced in Section 6.

2. Algebraic framework

Mathematical morphology [91] usually relies on the algebraic framework of complete lattices, which has been justified in particular in [86], since it allows dealing properly with functions taking values in a bounded interval (which is useful in the present context). It has also been extended to complete semi-lattices and general posets [66], based on the notion of adjunction [60] (see also [21] for a general description of the algebraic framework). In this paper, we only consider the case of complete lattices. We first introduce bipolar information models, and then a lattice structure on them, according to some partial ordering, which can be specified for any particular domain of application. Then bipolar connectives are defined. The presentation and notations are chosen to highlight the bipolar nature of the information to be processed. However, the mathematical framework is the same as in any complete lattice and relies on general results of this domain [13,37,53].

2.1. Bipolar information

As mentioned in the introduction, bipolar information has two components, one related to positive information, and one related to negative information. These pieces of information can take different forms, according to the application domain, such as preferences and constraints, observations and rules, possible and forbidden places for an object in space, etc.

Let us assume that bipolar information is represented by a pair (μ, ν), where μ represents the positive information and ν the negative information, under a consistency constraint [47], which guarantees that the positive information is compatible with the constraints or rules expressed by the negative information. From a formal point of view, bipolar information can be represented in different settings, depending on the application domain, leading to different forms of μ and ν, which are all mathematically equivalent. Let us mention for instance:

- Positive and negative information are subsets P and N of some set, and the consistency constraint is expressed as $P \cap N = \emptyset$, expressing that what is possible or preferred (positive information) should be included in what is not forbidden (negative information) [47].

- μ and ν are membership functions to fuzzy sets, defined over a space $\mathcal{S}$, and the consistency constraint is expressed as $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. The pair (μ, ν) is then called a bipolar fuzzy set. As noticed e.g., in [46,57], although there are important differences in semantics, bipolar fuzzy sets are formally equivalent to interval-valued fuzzy sets originally proposed in [103], where the membership of x is expressed (using the same notations) as an interval $[\mu(x), 1 - \nu(x)]$ of $[0, 1]$ (hence implying the consistency constraint), and to intuitionistic fuzzy sets, where this consistency constraint was also proposed, along with the notion of membership and non-membership degrees [6]. All these are also special cases of L-fuzzy sets [46].
- Positive and negative information are represented by logical formulas φ and ψ, generated by a set of propositional symbols and connectives, and the consistency constraint is then expressed as $\varphi \wedge \psi \models \perp$ (ψ represents what is forbidden or impossible). Examples of logical formalisms for handling bipolarity include [65,67].
- Other examples include functions such as utility functions or capacities [56], bi-capacities [68], preference functions [82], four-valued logics [67], possibility distributions [49,50,81].

In the following, we will detail the case of bipolar fuzzy sets, extending our previous work in [15,16,18,19] to any partial ordering. This case includes the other examples described above: the case of sets corresponds to the case where only bipolarity should be taken into account, without fuzziness (hence the membership function takes only values 0 and 1). In the case of logical formulas, we consider the models $[\![\varphi]\!]$ and $[\![\psi]\!]$ as sets or fuzzy sets. The lattice defined on the set of models is isomorphic to the one defined on $\Phi_{\equiv}$, where $\Phi_{\equiv}$ denotes the quotient space of the set of formulas Φ by the syntactic equivalence relation between formulas (defined as $\varphi \equiv \varphi'$ iff $[\![\varphi]\!] = [\![\varphi']\!]$). Hence the case of bipolar fuzzy sets is general enough to cover several other settings.

Let $\mathcal{S}$ be the underlying space (the spatial domain for spatial information processing for instance).

Definition 1. A bipolar fuzzy set on $\mathcal{S}$ is defined by an ordered pair of functions (μ, ν) from $\mathcal{S}$ into $[0, 1]$ such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$ (consistency constraint).

Although a bipolar fuzzy set is formally equivalent to an intuitionistic fuzzy set or to an interval-valued fuzzy set [6,103] with the same consistency constraint as the one proposed in these two domains (and the associated structures are isomorphic) [28,46,57], the semantics are very different, and we keep here the terminology of bipolarity. A discussion on semantics is proposed in Section 2.4. An important point is that we consider here that μ and ν are really two different functions, which may represent different types of information or may be issued from different sources. However, this may also include the symmetric case, reducing the consistency constraint to a duality relation such as $\nu = 1 - \mu$. The proposed approach also differs from the one in [104] where bipolarity is encoded on $[-1, 0] \times [0, 1]$ for defining bipolar fuzzy logic. For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ its non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. Since the positive information models what is possible, preferred, observed or experienced, and the negative information what is forbidden or impossible, the consistency constraint avoids contradictions between what is forbidden and what is possible (i.e., the potential solutions should be included in what is not forbidden or impossible). The set of bipolar fuzzy sets defined on $\mathcal{S}$ is denoted by $\mathcal{B}$.

Let us denote by $\mathcal{L}$ the set of ordered pairs of numbers (a, b) in $[0, 1]$ such that $a + b \leq 1$ (hence $(\mu, \nu) \in \mathcal{B} \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x), \nu(x)) \in \mathcal{L}$). In all what follows, for each $(\mu, \nu) \in \mathcal{B}$, we will note $(\mu, \nu)(x) = (\mu(x), \nu(x))$ $(\in \mathcal{L})$, $\forall x \in \mathcal{S}$. Note that fuzzy sets can be considered as particular cases of bipolar fuzzy sets, either when $\forall x \in \mathcal{S}, \nu(x) = 1 - \mu(x)$, or when only one information is available, i.e., $(\mu(x), 0)$ or $(0, 1 - \mu(x))$. Furthermore, if μ (and ν) only takes values 0 and 1, then bipolar fuzzy sets reduce to classical sets. Let us finally comment about the choice of $\mathcal{L}$. A more general setting could rely on L-fuzzy sets [55], by only assuming that $\mathcal{L}$ is a poset or a complete lattice. This line was followed in the recent work [97] for instance. Here we have chosen to keep $\mathcal{L}$ as presented above for simplifying the presentation and highlighting the bipolar nature of the information with its two components, but all what follows in this section and in the next one actually applies to more general forms of it. An important point is that we keep the partial ordering on $\mathcal{L}$ as a free parameter (see next), which is one of the main focus of this paper, and may have an important impact on applications.

2.2. Partial ordering and lattice of bipolar fuzzy sets

Let $\preceq$ be a partial ordering on $\mathcal{L}$ such that $(\mathcal{L}, \preceq)$ is a complete lattice. We denote by $\bigvee$ and $\bigwedge$ the supremum and infimum, respectively. The smallest element is denoted by $0_{\mathcal{L}}$ and the largest element by $1_{\mathcal{L}}$. We denote by $\succeq$ the reverse order, i.e., $\forall((a, b), (a', b')) \in \mathcal{L}^2, (a, b) \succeq (a', b') \Leftrightarrow (a', b') \preceq (a, b)$.

The partial ordering on $\mathcal{L}$ induces a partial ordering on $\mathcal{B}$, also denoted by $\preceq$ for the sake of simplicity:

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \text{ iff } \forall x \in \mathcal{S}, (\mu_1(x), \nu_1(x)) \preceq (\mu_2(x), \nu_2(x)). \tag{1}$$

Then $(\mathcal{B}, \preceq)$ is a complete lattice, for which the supremum and infimum are also denoted by $\bigvee$ and $\bigwedge$. The smallest element is the bipolar fuzzy set (μ_0, ν_0) taking value $0_{\mathcal{L}}$ at each point, and the largest element is the bipolar fuzzy set $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ always equal to $1_{\mathcal{L}}$.

The following result, which holds in any lattice [13], is useful for the results in the next sections:

$$\forall (a_1, b_1) \in \mathcal{L}, \forall (a_2, b_2) \in \mathcal{L}, \ (a_1, b_1) \preceq (a_2, b_2) \Leftrightarrow \begin{cases} (a_1, b_1) \vee (a_2, b_2) = (a_2, b_2) \\ (a_1, b_1) \wedge (a_2, b_2) = (a_1, b_1) \end{cases}$$

and similarly in $\mathcal{B}$:

$$\forall (\mu, \nu) \in \mathcal{B}, \forall (\mu', \nu') \in \mathcal{B}, \ (\mu, \nu) \preceq (\mu', \nu') \Leftrightarrow \begin{cases} (\mu, \nu) \vee (\mu', \nu') = (\mu', \nu') \\ (\mu, \nu) \wedge (\mu', \nu') = (\mu, \nu) \end{cases}$$

Note that the supremum and the infimum do not necessarily provide one of the input bipolar numbers or bipolar fuzzy sets (in particular if they are not comparable according to $\preceq$). However, they do if $\preceq$ is a total ordering.

2.3. Bipolar connectives

Let us now recall definitions and properties of connectives, that will be useful in the following and that extend to the bipolar case the connectives classically used in fuzzy set theory [19]. In all what follows, increasingness and decreasingness are intended according to the partial ordering $\preceq$. Similar definitions can also be found e.g. in [34,42,40] in the case of interval-valued fuzzy sets of intuitionistic fuzzy sets, for a specific partial ordering (Pareto-like ordering), or more recently in [97] in the more general setting of L-fuzzy sets [55].

Definition 2. A **negation, or complementation**, on $\mathcal{L}$ is a decreasing operator N such that $N(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $N(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. In this paper, we restrict ourselves to involutive negations, such that $\forall \mathbf{a} \in \mathcal{L}, N(N(\mathbf{a})) = \mathbf{a}$ (these are the most interesting ones for mathematical morphology).
A **conjunction** is an operator C from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $C(0_{\mathcal{L}}, 0_{\mathcal{L}}) = C(0_{\mathcal{L}}, 1_{\mathcal{L}}) = C(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}, C(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$, and that is increasing in both arguments.[1]
A **t-norm** is a commutative and associative bipolar conjunction such that $\forall \mathbf{a} \in \mathcal{L}, C(\mathbf{a}, 1_{\mathcal{L}}) = C(1_{\mathcal{L}}, \mathbf{a}) = \mathbf{a}$ (i.e., the largest element of $\mathcal{L}$ is the unit element of C). If only the property on the unit element holds, then C is called a **semi-norm**.
A **disjunction** is an operator D from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $D(1_{\mathcal{L}}, 1_{\mathcal{L}}) = D(0_{\mathcal{L}}, 1_{\mathcal{L}}) = D(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 1_{\mathcal{L}}, D(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$, and that is increasing in both arguments.
A **t-conorm** is a commutative and associative bipolar disjunction such that $\forall \mathbf{a} \in \mathcal{L}, D(\mathbf{a}, 0_{\mathcal{L}}) = D(0_{\mathcal{L}}, \mathbf{a}) = \mathbf{a}$ (i.e., the smallest element of $\mathcal{L}$ is the unit element of D).
An **implication** is an operator I from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $I(0_{\mathcal{L}}, 0_{\mathcal{L}}) = I(0_{\mathcal{L}}, 1_{\mathcal{L}}) = I(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}, I(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$ and that is decreasing in the first argument and increasing in the second argument.

In the following, we will call these connectives **bipolar** to make their instantiation on bipolar information explicit. Similarly, elements of $\mathcal{L}$ should be considered as pairs, quantifying the negative and positive parts of information.

A number of properties directly follow from the definitions.

- Bipolar connectives reduce to classical fuzzy connectives in the limit cases where there is no bipolarity in the input value and in the result. Let C be a bipolar t-norm. Then, under the non-bipolarity conditions, there exists a t-norm t such that $\forall (a_1, a_2) \in [0, 1]^2, \ C((a_1, 1 - a_1), (a_2, 1 - a_2)) = (t(a_1, a_2), 1 - t(a_1, a_2))$, or $C((a_1, 0), (a_2, 0)) = (t(a_1, a_2), 0)$ for the embedding of fuzzy sets in $\mathcal{B}$ as $(\mu, 0)$. Similar expressions hold for the other connectives.
- Any bipolar conjunction C has a null element, which is the smallest element of $\mathcal{L}$: $\forall \mathbf{a} \in \mathcal{L}, C(\mathbf{a}, 0_{\mathcal{L}}) = C(0_{\mathcal{L}}, \mathbf{a}) = 0_{\mathcal{L}}$.
- Similarly, any bipolar disjunction has a null element, which is the largest element of $\mathcal{L}$: $\forall \mathbf{a} \in \mathcal{L}, D(\mathbf{a}, 1_{\mathcal{L}}) = D(1_{\mathcal{L}}, \mathbf{a}) = 1_{\mathcal{L}}$.
- For implications, we have $\forall \mathbf{a} \in \mathcal{L}, I(0_{\mathcal{L}}, \mathbf{a}) = I(\mathbf{a}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$.

As in the fuzzy case, conjunctions and implications may be related to each other based on the residuation principle, which corresponds to a notion of adjunction, which is also fundamental in mathematical morphology. This principle is expressed as follows in the bipolar case.

Definition 3. A pair of bipolar connectives (I, C) forms an adjunction if: $\forall (a_i, b_i) \in \mathcal{L}, i = 1, \ldots, 3$,

$$C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2) \Leftrightarrow (a_3, b_3) \preceq I((a_1, b_1), (a_2, b_2)). \tag{2}$$

The connectives in Definition 2 can be linked to each other in different ways (again this is similar to the fuzzy case).

[1] That is, $\forall (a_1, a_2, a'_1, a'_2) \in \mathcal{L}^4, a_1 \preceq a'_1$ and $a_2 \preceq a'_2 \Rightarrow C(a_1, a_2) \preceq C(a'_1, a'_2)$.

Proposition 1. *The following properties hold for bipolar connectives:*

- *Given a t-norm C and a negation N, the following operator D defines a t-conorm:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$D((a_1, b_1), (a_2, b_2)) = N(C(N((a_1, b_1)), N((a_2, b_2)))). \quad (3)$$
- *An implication I induces a negation N defined as*:
$$\forall(a, b) \in \mathcal{L}, N((a, b)) = I((a, b), 0_{\mathcal{L}}). \quad (4)$$
- *The following operator* I_N*, derived from a negation N and a conjunction C, defines an implication:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$I_N((a_1, b_1), (a_2, b_2)) = N(C((a_1, b_1), N((a_2, b_2)))). \quad (5)$$
- *Conversely, a conjunction C can be derived from a negation N and an implication I:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$C((a_1, b_1), (a_2, b_2)) = N(I((a_1, b_1), N((a_2, b_2)))). \quad (6)$$
- *Similarly, an implication can be derived from a negation N and a disjunction D as:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$I_N((a_1, b_1), (a_2, b_2)) = D(N((a_1, b_1)), (a_2, b_2)). \quad (7)$$
- *An implication can also be defined by residuation from a conjunction C such that* $\forall(a, b) \in \mathcal{L} \setminus 0_{\mathcal{L}}, C(1_{\mathcal{L}}, (a, b)) \neq 0_{\mathcal{L}}$ *(leading to an adjunction* (C, I_R)*, see Definition 3):* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$I_R((a_1, b_1), (a_2, b_2)) = \bigvee\{(a_3, b_3) \in \mathcal{L} \mid C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2)\}. \quad (8)$$
- *Conversely, from an implication* I_R *such that* $\forall(a, b) \in \mathcal{L} \setminus 1_{\mathcal{L}}, I_R(1_{\mathcal{L}}, (a, b)) \neq 1_{\mathcal{L}}$*, the conjunction C such that* (C, I_R) *forms an adjunction is given by:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,
$$C((a_1, b_1), (a_2, b_2)) = \bigwedge\{(a_3, b_3) \in \mathcal{L} \mid (a_2, b_2) \preceq I_R((a_1, b_1), (a_3, b_3))\}. \quad (9)$$

Proposition 2. *Let C be a bipolar conjunction and I a bipolar implication derived from C, either as* I_N *using an involutive negation (Eq. (5)) or as* I_R *by residuation (Eq. (8)). The following equivalence holds:*
$$\forall(a, b) \in \mathcal{L}, C(1_{\mathcal{L}}, (a, b)) = (a, b) \Leftrightarrow \forall(a, b) \in \mathcal{L}, I(1_{\mathcal{L}}, (a, b)) = (a, b), \quad (10)$$
i.e., C admits $1_{\mathcal{L}}$ *as unit element on the left iff I admits* $1_{\mathcal{L}}$ *as unit element on the left.*

This result directly follows from Eqs. (5), (6), (8) and (9).

Proposition 3. *If C and I are bipolar connectives such that* (I, C) *forms an adjunction (i.e., verifies Eq. (2)), then C distributes over the supremum and I over the infimum on the right, i.e.:* $\forall(a_i, b_i) \in \mathcal{L}, \forall(a, b) \in \mathcal{L}$,
$$\bigvee_i C((a, b), (a_i, b_i)) = C\left((a, b), \bigvee_i (a_i, b_i)\right), \quad (11)$$
$$\bigwedge_i I((a, b), (a_i, b_i)) = I\left((a, b), \bigwedge_i (a_i, b_i)\right). \quad (12)$$

The proof is similar to the classical proof done in mathematical morphology to show that an adjunction defines a dilation (i.e., an operation that commutes with the supremum) and an erosion [59].

Note that the distributivity on the left requires C to be commutative, and in that case we also have:
$$\bigvee_i C((a_i, b_i), (a, b)) = C\left(\bigvee_i (a_i, b_i), (a, b)\right), \quad (13)$$
and then we have in a similar way for I:
$$\bigwedge_i I((a_i, b_i), (a, b)) = I\left(\bigvee_i (a_i, b_i), (a, b)\right). \quad (14)$$

The following properties of adjunctions will also be useful for deriving mathematical morphology operators.

Proposition 4. *Let* (I, C) *be an adjunction. Then the following properties hold:*

- *C is increasing in the second argument and I in the second one. If furthermore C is commutative, then it is also increasing in the first one.*
- $0_{\mathcal{L}}$ *is the null element of C on the right and* $1_{\mathcal{L}}$ *is the null element of I on the right, i.e.,* $\forall(a, b) \in \mathcal{L}, C((a, b), 0_{\mathcal{L}}) = 0_{\mathcal{L}},\ I((a, b), 1_{\mathcal{L}}) = 1_{\mathcal{L}}$.

Again this follows from classical results in mathematical morphology, by noting that $C((a, b), .)$ and $I((a, b), .)$ are respectively a dilation and an erosion, for every (a, b).

Finally, some ordering properties hold with respect to the infimum and the supremum of the lattice $(\mathcal{L}, \preceq)$. More ordering properties can be exhibited for specific orderings, as we will see later on for the Pareto ordering.

Proposition 5.

- *Let C be a conjunction that admits* $1_{\mathcal{L}}$ *as unit element. Then* $\forall((a, b), (a', b')) \in \mathcal{L}^2,\ C((a, b), (a', b')) \preceq (a, b) \wedge (a', b')$.
- *Let I be an implication that admits* $1_{\mathcal{L}}$ *as unit element on the left. Then* $\forall((a, b), (a', b')) \in \mathcal{L}^2,\ (a', b') \preceq I((a, b), (a', b'))$.
- *Let I be an implication that admits* $0_{\mathcal{L}}$ *as unit element on the right. Then* $\forall((a, b), (a', b')) \in \mathcal{L}^2,\ (a, b) \preceq I((a, b), (a', b'))$.

It follows that, since $\wedge$ *is a bipolar t-norm (from standard properties of lattices), it is the largest conjunction having* $1_{\mathcal{L}}$ *as unit element. It is moreover idempotent. Similarly* $\vee$ *is a bipolar t-conorm and is the smallest disjunction having* $0_{\mathcal{L}}$ *as unit element. It is also idempotent.*

2.4. A few comments about semantics

It is interesting to note that bipolar fuzzy sets are formally linked to intuitionistic fuzzy sets [6], interval-valued fuzzy sets [103] and vague sets, or to clouds when boundary constraints are added [43,80], as shown by several authors [28,46]. Equivalences and redundancies have been extensively discussed in [57]. However, their respective semantics differ, and, as mentioned in [46], this is a major point to be taken into account, beyond mathematical equivalence, with important impact on applications. Concerning intuitionistic fuzzy sets, there have been many discussions about terminology in this domain recently [7,46,57], mainly because of the word "intuitionistic" which is misleading and introduces confusions with intuitionistic logics. The semantics of intuitionistic fuzzy sets reveals the possibility of some indetermination between membership and non-membership to a set. As for interval-valued fuzzy sets or clouds, their semantics correspond to the representation of some imprecision or uncertainty about the membership value, which can then only be given as an interval, and not as a crisp number.

However the semantics of bipolar fuzzy sets is different. A bipolar fuzzy set in the spatial domain does not necessarily represent one physical object or spatial entity, but twofold information about it, potentially issued from different sources. It can be two different regions of space, conveying information on a physical object. For instance the region representing the positive information can inform on how or where the object is or could be, while the region representing the negative information can put some constraints or define forbidden places for the object. This refers to the third type of bipolarity, according to the classification presented in [47,50]. An example is the modeling of information concerning the location of a robot: positive information could concern potential locations (for instance derived from sensor data) and negative information could concern forbidden places (because they are already occupied by other objects, or the robot is not allowed to move there because there are some fragile objects that should not be touched, etc.). What is then represented is not a model of uncertainty in the standard meaning since it is not just an uncertainty at each position about the possible location of the robot at this position. This differs from what is done using interval-valued fuzzy sets. For other types of bipolarity (symmetric uni- or bi-variate), a bipolar fuzzy set could be one physical object, imperfectly known. This is also the case in interval-valued fuzzy sets modeling, which is until now the most widely used in image processing applications (see also Section 4.4). The kind of bipolarity depends on the type of information to be handled and on the applications.

Let us suggest another example in medical imaging, where an image-guided radiotherapy of a tumor has to be performed. Image information provides positive information about the position of the tumor. Some spatial imprecision (due to the tumor itself, the part of it actually seen in the image, the segmentation process, the margin introduced for radiotherapy, ...) can be modeled using a fuzzy dilation. Negative information may come from a constraint, such as avoiding organs at risk that should not undergo radiations (heart for instance). This information is not directly related to the tumor and is of very different nature. Therefore the bipolar information cannot be simply considered as an uncertainty about the tumor. In this example, the constraint should be considered in a strict way. However this is not always the case, depending on the applications. For instance if the negative information represents some generic anatomical knowledge, observations of a specific patient can deviate from the normal case, and should then have the priority. Although asymmetric bipolarity is considered in both examples, the two types of information should not be handled similarly in both cases. This can be achieved by an appropriate choice of the partial ordering, as discussed next.

Let us consider a different domain, about preference modeling [9]. Positive information describes what is desired and allows sorting solutions, while negative information describes what is rejected or unacceptable and defines constraints to be fulfilled. The gap between positive and negative information does not necessarily concern indetermination, but rather neutrality or indifference. Let us consider as an example preferences of agents about the countries in which they would like to travel, and constraints about their travels, represented in a logical formalism, as briefly introduced in Section 2.1. The set of propositional symbols if the set of all countries in the world. Preferences are denoted by formulas φ and constraints by formulas ψ. In the following example, we show how dilation of bipolar representations of preferences and constraints can help reaching an agreement between agents. Let us assume that Agent 1: (i) prefers to travel in Spain: $\varphi_1 = Spain$, (ii) has to stay in Europe: $\psi_1 = \neg(Belgium \vee France \vee Spain \vee Portugal \vee Italy \vee Germany \vee TheNetherlands \vee \ldots\}$. On the other hand, Agent 2: (i) prefers to travel in Morocco: $\varphi_2 = Morocco$, (ii) has to stay in a Mediterranean country: $\psi_2 = \neg(Morocco \vee Spain \vee Italy \vee Portugal \vee \ldots)$. In this example, each agent is consistent but the two agents have conflicting preferences. However, each agent is now ready to extend his preferences so that the two agents can travel together (under the conditions that the constraints, which are fixed, are satisfied). This can be simply modeled by a dilation δ, as presented in the next sections, such that some neighbor countries are included in the preferences, conditioned by the constraints:

$$\delta(\varphi_1) = Spain \vee France \vee Portugal \vee Morocco$$

$$\delta(\varphi_2) = Morocco \vee Algeria \vee Portugal \vee Spain$$

Introducing the constraints in order to satisfy the consistency requirements leads to:

$$\varphi_1' = \delta(\varphi_1) \wedge \psi_1 = Spain \vee France \vee Portugal$$

$$\varphi_2' = \delta(\varphi_2) \wedge \psi_2 = \delta(\varphi_2)$$

Now the preferences are no more conflicting. The fusion of the agents' preferences and constraints can be expressed as the conjunction of the preferences and disjunction of the constraints:

$$(\varphi, \psi) = (\varphi_1' \wedge \varphi_2', \psi_1 \vee \psi_2) = \left(Spain \vee Portugal, \neg\left(\bigvee Medit.\ and\ Eur.\ countries\right)\right)$$

A solution for travelling can then be found in the set of models of these formulas. Although ψ could also be eroded so as to reduce the constraints, it is not in this simple example. This corresponds to a situation where constraints are strict and cannot be violated. Another situation could be that constraints can be considered with some flexibility, and in that case ψ would be eroded.

2.5. A note on partial ordering

One of the main issues in the proposed extensions of mathematical morphology to bipolar information is to handle the two components (i.e., positive and negative information) and to define an adequate and relevant ordering. Two extreme cases are Pareto ordering (also called marginal ordering) and lexicographic ordering. The Pareto ordering handles both components in a symmetric way, while the lexicographic ordering on the contrary gives a strong priority to one component, and the other one is then seldom considered. These features can be seen as either advantages or drawbacks, depending on the context and on the application, and will be further discussed in Sections 4.5 and 5.5. Roughly speaking, Pareto ordering seems adapted to symmetric bipolarity (according to the classification proposed in [50]), while lexicographic ordering can handle asymmetric bipolarity. However, this may vary depending on the application and on the information to be processed, and there may be situations where negative and positive information coming from different sources and not directly related to each other could be handled in a symmetric way.

This issue has been addressed in other types of work, where different partial orderings have been discussed. We mention here two examples: color image processing and social choice.

Color image processing: The question of defining a suitable ordering on vectorial images (in particular color images) has been widely addressed in the mathematical imaging community (see e.g., [4] for a review). The lexicographic ordering is known to excessively privilege one of the colors, and can be refined for instance by defining a rougher quantization on the first color, so as to have more frequent comparisons based on the two other ones. Other approaches use a 1D scale by applying a scalar function to the vectors, define groups of vectors which are then ranked, or base the comparison on a distance to a reference vector, just to cite a few ones [3–5].

Social choice: This is another domain where the question of defining a partial ordering is crucial, in particular for multi-criteria decision making or voting problems. Various orderings have been proposed, including refinements of the lexicographic ordering, leximin/leximax, discrimax, tolerant Pareto, etc. [25,45,51,76,89]. An axiomatization of qualitative comparison has been proposed in [44] for handling positive and negative features, and several rules have been defined and compared.

All these works can guide the choice of an ordering adapted to bipolar information.

The following section remains general, and applies to any partial ordering, while two specific examples will be detailed next: Pareto ordering in Section 4 and lexicographic ordering in Section 5.

3. Dilations and erosions of bipolar fuzzy sets

3.1. General algebraic case

Once we have a complete lattice, it is easy to define algebraic dilations and erosions on this lattice, as classically done in mathematical morphology [59,60,92]. Here we only consider operations from the lattice $(\mathcal{B}, \preceq)$ into itself. Proofs are omitted in this section, since they are exactly the same as in any complete lattice, and there is nothing specific to do for the particular case of the lattice $(\mathcal{B}, \preceq)$. The definitions and properties are just given, using the notations of bipolar fuzzy sets to highlight the role of the two components, for the sake of completeness.

A dilation is an operator δ from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the supremum:

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \delta\left(\bigvee_i (\mu_i, \nu_i)\right) = \bigvee_i \delta((\mu_i, \nu_i)), \tag{15}$$

where (μ_i, ν_i) is any family (finite or not) of elements of $\mathcal{B}$.
An erosion is an operator ε from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the infimum:

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \varepsilon(\bigwedge_i (\mu_i, \nu_i)) = \bigwedge_i \varepsilon((\mu_i, \nu_i)). \tag{16}$$

Algebraic dilations δ and erosions ε on $\mathcal{B}$ satisfy the following properties:

- δ and ε are increasing operators;
- δ preserves the smallest element: $\delta((\mu_0, \nu_0)) = (\mu_0, \nu_0)$;
- ε preserves the largest element: $\varepsilon((\mu_{\mathbb{I}}, \nu_{\mathbb{I}})) = (\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$;
- by denoting (μ_x, ν_x) the canonical bipolar fuzzy set associated with (μ, ν) and x such that $(\mu_x, \nu_x)(x) = (\mu(x), \nu(x))$ and $\forall y \in \mathcal{S} \setminus \{x\}$, $(\mu_x, \nu_x)(y) = 0_{\mathcal{L}}$, we have $(\mu, \nu) = \bigvee_x (\mu_x, \nu_x)$ and $\delta((\mu, \nu)) = \bigvee_x \delta((\mu_x, \nu_x))$.

The last result leads to morphological operators in case $\delta((\mu_x, \nu_x))$ has the same "shape" everywhere (and is then a bipolar fuzzy structuring element). This case is detailed in Section 3.2.

A fundamental notion in this algebraic framework is the one of adjunction. A pair of operators (ε, δ) defines an adjunction on $(\mathcal{B}, \preceq)$ iff:

$$\forall (\mu, \nu) \in \mathcal{B}, \forall (\mu', \nu') \in \mathcal{B},\ \delta((\mu, \nu)) \preceq (\mu', \nu') \Leftrightarrow (\mu, \nu) \preceq \varepsilon((\mu', \nu')) \tag{17}$$

If a pair of operators (ε, δ) on $\mathcal{B}$ defines an adjunction, then the following results hold:

- δ preserves the smallest element and ε the largest element of the lattice;
- δ is a dilation and ε is an erosion, in the sense of Eqs. (15) and (16);
- $\delta\varepsilon$ is anti-extensive: $\delta\varepsilon \preceq Id$, where Id denotes the identity mapping on $\mathcal{B}$ (i.e., $\forall (\mu, \nu) \in \mathcal{B}, Id(\mu, \nu) = (\mu, \nu)$), and $\varepsilon\delta$ is extensive: $Id \preceq \varepsilon\delta$ (the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are called morphological opening and morphological closing, respectively);
- $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ and $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$, i.e., morphological opening and closing are idempotent operators.

Let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive and $\varepsilon\delta$ is extensive. Then (ε, δ) is an adjunction. The following representation result also holds. If ε is an increasing operator, it is an algebraic erosion if and only if there exists δ such that (ε, δ) is an adjunction. The operator δ is then an algebraic dilation and can be expressed as:

$$\delta((\mu, \nu)) = \bigwedge\{(\mu', \nu') \in \mathcal{B} \mid (\mu, \nu) \preceq \varepsilon((\mu', \nu'))\}. \tag{18}$$

A similar representation result holds for erosion.

3.2. Morphological dilations and erosions of bipolar fuzzy sets

Particular forms of dilations and erosions, called morphological dilations and erosions, are defined in classical morphology, involving the notion of structuring element [91]. In the spatial domain $\mathcal{S}$ for instance ($\mathcal{S}$ is then assumed to be an affine space or at least a space where translations can be defined), a structuring element is a subset of $\mathcal{S}$ with fixed shape and size, directly influencing the spatial extent of the morphological transformations. It is generally assumed to be compact, so as to guarantee good properties. In the discrete case, it is often assumed to be connected, in the sense of a discrete connectivity defined on $\mathcal{S}$. The general principle underlying morphological operators, under an assumption of invariance by translation, consists in translating the structuring element at every position in space and checking if this translated structuring element satisfies some relation with the original set (inclusion for erosion, intersection for dilation) [59,60,88,91,92]. This principle has also been used in the main extensions of mathematical morphology to fuzzy sets [22,38,39,71,77,94]. It has been further investigated in the algebraic framework of quantales [1,90,96]. More generally, without any assumption on the underlying domain $\mathcal{S}$, a structuring element is defined as a binary relation between two elements of $\mathcal{S}$ (i.e., y is in relation with x if and only if $y \in B_x$) [21]. This allows on the one hand dealing with spatially varying structuring elements (when $\mathcal{S}$ is the spatial domain), as e.g. in [11,24], or with graph structures (e.g., [100]), and on the other hand establishing interesting links with several other domains, such as rough sets, formal logics, and, in the more general case where the morphological operations are defined from one set to another one, with Galois connections and formal concept analysis, as shown e.g. in [21].

From now on, we assume that $\mathcal{S}$ is an affine space on which translations are defined (but all definitions and results also apply to the other situations mentioned above). Following the same principle as in classical morphology, defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets and interval-valued fuzzy sets [34,42]. With the notations adopted here, a degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as [15]:

$$\bigwedge_{x\in\mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \quad (19)$$

where I is a bipolar implication, and a degree of intersection is defined as:

$$\bigvee_{x\in\mathcal{S}} C((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \quad (20)$$

where C is a bipolar conjunction. Note that both inclusion and intersection degrees are elements of $\mathcal{L}$, i.e., they are defined as bipolar degrees.

Based on these concepts, we can now propose a general definition for morphological erosions and dilations, thus extending our previous work in [15,16,19].

Definition 4. Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The erosion of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar implication I as:

$$\forall x \in \mathcal{S},\ \ \varepsilon_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigwedge_{y\in\mathcal{S}} I((\mu_B(y-x), \nu_B(y-x)), (\mu(y), \nu(y))). \quad (21)$$

In this equation, $\mu_B(y-x)$ (respectively $\nu_B(y-x)$) represents the value at point y of the translation of μ_B (respectively ν_B) at point x.

Definition 5. Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The dilation of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar conjunction C as:

$$\delta_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigvee_{y\in\mathcal{S}} C((\mu_B(x-y), \nu_B(x-y)), (\mu(y), \nu(y))). \quad (22)$$

Definitions 4 and 5 are consistent: they actually provide bipolar fuzzy sets of $\mathcal{B}$, i.e., $\forall(\mu, \nu) \in \mathcal{B}$, $\forall(\mu_B, \nu_B) \in \mathcal{B}$, $\delta_{(\mu_B,\nu_B)}((\mu, \nu)) \in \mathcal{B}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \in \mathcal{B}$.

In the particular case where bipolar fuzzy sets are usual fuzzy sets (i.e., $\nu = 1 - \mu$ and $\nu_B = 1 - \mu_B$, or $\nu = 0$ and $\nu_B = 0$), the definitions lead to the usual definitions of fuzzy dilations and erosions. Hence they are also compatible with classical morphology in the case where μ and μ_B are crisp.

Proposition 6. *Definitions 4 and 5 provide an adjunction (ε, δ) if and only if (I, C) is an adjunction.*

It follows that if I and C are bipolar connectives such that (I, C) is an adjunction, then the operator ε defined from I by Eq. (21) commutes with the infimum and the operator δ defined from C by Eq. (22) commutes with the supremum, i.e., they are algebraic erosion and dilation. Moreover they are increasing with respect to (μ, ν).

Proposition 7. *If (I, C) is an adjunction such that C is increasing in the first argument and I is decreasing in the first argument (typically if they are a bipolar conjunction and a bipolar implication), then the operator ε defined from I by Eq. (21) is decreasing with respect to the bipolar fuzzy structuring element and the operator δ defined from C by Eq. (22) is increasing with respect to the bipolar fuzzy structuring element.*

These monotony properties are directly derived from the ones of $C, I, \vee$ and $\wedge$.

Proposition 8. *C distributes over the supremum and I over the infimum on the right if and only if ε and δ defined by Eqs. (21) and (22) are algebraic erosion and dilation, respectively.*

In the following, we only consider cases where the definitions actually provide algebraic dilations and erosions (which are the only ones that are interesting). Obviously, all results of Section 3.1 also hold.

Note that while δ commutes with the supremum and ε with the infimum, the converse is generally not true. However, inequalities hold, as in classical morphology.

Proposition 9. *Let δ and ε be a dilation and an erosion defined by Eqs. (22) and (21). Then, for all $(\mu_B, \nu_B), (\mu, \nu), (\mu', \nu')$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)}((\mu, \nu) \wedge (\mu', \nu')) \preceq \delta_{(\mu_B,\nu_B)}((\mu, \nu)) \wedge \delta_{(\mu_B,\nu_B)}((\mu', \nu')), \tag{23}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \vee \varepsilon_{(\mu_B,\nu_B)}((\mu', \nu')) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu, \nu) \vee (\mu', \nu')). \tag{24}$$

These results are derived from the increasingness of C and the increasingness of I with respect to the second argument.

Proposition 10. *A dilation δ defined by Eq. (22) is increasing with respect to the bipolar fuzzy structuring element, while an erosion ε defined by Eq. (21) is decreasing with respect to the bipolar fuzzy structuring element.*

These results are directly derived from the increasingness of C, $\vee$, $\wedge$ and from the decreasingness of I with respect to the first argument.

These results fit well with the intuitive meaning behind the morphological operators. Indeed, a dilation is interpreted as a degree of intersection, which is easier to achieve with a larger structuring element, while an erosion is interpreted as a degree of inclusion, which means a stronger constraint if the structuring element is larger.

Proposition 11. *Let δ and ε be a dilation and an erosion defined by Eqs. (22) and (21). Then, for all $(\mu_B, \nu_B), (\mu'_B, \nu'_B), (\mu, \nu)$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu, \nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu, \nu)) \wedge \delta_{(\mu'_B,\nu'_B)}((\mu, \nu)), \tag{25}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \vee \varepsilon_{(\mu'_B,\nu'_B)}((\mu, \nu)) \preceq \varepsilon_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu, \nu)). \tag{26}$$

These results are derived from the increasingness of C and the decreasingness of I with respect to the first argument.

Depending on the choice of C and I, some additional properties may hold.

Proposition 12. *Let δ be a dilation defined by Eq. (22) from a bipolar conjunction C. The dilation satisfies $\delta_{(\mu_B,\nu_B)}((\mu, \nu)) = \delta_{(\mu,\nu)}((\mu_B, \nu_B))$ if and only if C is commutative.*

This result is quite intuitive. When interpreting the dilation as a degree of intersection, it is natural to expect this degree to be symmetrical in both arguments. Hence the commutativity of C has to be satisfied.

Proposition 13. *Let δ be a dilation defined by Eq. (22) from a bipolar conjunction C. It satisfies the iterativity property, i.e., by denoting δ_1 the dilation with structuring element (μ_B, ν_B), by δ_2 the dilation with structuring element (μ'_B, ν'_B) and by δ_{1-2} the dilation with structuring element $\delta_1(\mu'_B, \nu'_B)$, we have: $\delta_1(\delta_2(\mu, \nu)) = \delta_{1-2}((\mu, \nu))$ if and only if C is associative.*

Proposition 14. *Let δ be a dilation defined by Eq. (22) from a bipolar conjunction C. If C is a bipolar conjunction that admits $1_\mathcal{L}$ as unit element on the left (i.e., $\forall (a, b) \in \mathcal{L}$, $C(1_\mathcal{L}, (a, b)) = (a, b)$) and $C((a, b), 1_\mathcal{L}) \neq 1_\mathcal{L}$ for $(a, b) \neq 1_\mathcal{L}$, then the dilation is extensive, i.e., $\delta_{(\mu_B,\nu_B)}((\mu, \nu)) \succeq (\mu, \nu)$, if and only if $(\mu_B, \nu_B)(0) = 1_\mathcal{L}$, where 0 denotes the origin of space $\mathcal{S}$.*

A similar property holds for erosion and if I is a bipolar implication that admits $1_\mathcal{L}$ as unit element on the left (i.e., $\forall (a, b) \in \mathcal{L}$, $I(1_\mathcal{L}, (a, b)) = (a, b)$) and $I((a, b), 0_\mathcal{L}) \neq 0_\mathcal{L}$ for $(a, b) \neq 1_\mathcal{L}$, then the erosion is anti-extensive, i.e., $\varepsilon_{(\mu_B,\nu_B)}((\mu, \nu)) \preceq (\mu, \nu)$, if and only if $(\mu_B, \nu_B)(0) = 1_\mathcal{L}$.

The second condition on C holds in particular if $1_\mathcal{L}$ is also unit element on the right. This holds in specific cases in which C is a bipolar t-norm, which are the most interesting ones from a morphological point of view, as shown below.

Note that the condition $(\mu_B, \nu_B)(0) = 1_\mathcal{L}$ (i.e., the origin of space completely belongs to the bipolar fuzzy set, without any indetermination) is equivalent to the conditions on the structuring element found in classical [91] and fuzzy [22] morphology to have extensive dilations and anti-extensive erosions.

Proposition 15. *If I is derived from C and a negation N, then δ and ε are dual operators, i.e.: $\delta_{(\mu_B,\nu_B)}(N(\mu, \nu)) = N(\varepsilon_{(\check{\mu}_B,\check{\nu}_B)}((\mu, \nu)))$, where $(\check{\mu}_B, \check{\nu}_B)$ denotes the symmetrical bipolar fuzzy set of (μ_B, ν_B) with respect to the origin of $\mathcal{S}$.*

This result directly follows from Eq. (5).

Duality with respect to complementation, which was advocated in the first developments of mathematical morphology [91], is important to handle in an consistent way an object and its complement for many applications (for instance in image processing and spatial reasoning). Therefore it is useful to know exactly under which conditions this property may hold, so as to choose the appropriate operators if it is needed for a specific problem. On the other hand, adjunction is a major feature of the "modern" view of mathematical morphology, with strong algebraic bases in the framework of complete lattices [86]. This framework is now widely considered as the most interesting one, since it provides consistent definitions with sound properties in different settings (continuous and discrete ones) and mathematical morphology on bipolar fuzzy sets in this framework inherits a set of powerful and important properties. Due to the interesting features of these two properties of duality and adjunction, in several applications both are required.

From all these results, we can derive the following theorem, which shows that the proposed forms are the most general ones for C being a bipolar t-norm.

Theorem 1. *Definition 5 defines a dilation with all properties of classical mathematical morphology if and only if C is a bipolar t-norm. The adjoint erosion is then defined by Eq. (4) from the residual implication I_R derived from C. If the duality property is additionally required, then C and I have also to be dual operators with respect to a negation N.*

This theorem directly follows from the previous propositions.

This important result shows that taking any conjunction may not lead to dilations that have nice properties. For instance the iterativity of dilation is of prime importance in concrete applications, and it requires associative conjunctions. This is actually a main contribution of our work and the result is stronger and more general than previous ones in [15,16,19,73] since it applies for any partial ordering leading to a complete lattice on $\mathcal{B}$.

Note that pairs of adjoint operators are not necessarily dual. Therefore requiring both adjunction and duality properties may drastically reduce the choice for C and I. This will be illustrated for $\preceq$ being the Pareto partial ordering in Section 4. Note that this strong constraint is similar to the one proved for fuzzy sets in [17].

Although the choice of C and I is limited by the results expressed in Theorem 1 if sufficiently strong properties are required for the morphological operators, some choice may remain. The following property, derived from the monotony of the supremum and infimum, expresses a monotony property with respect to this choice.

Proposition 16. *Dilations and erosions are monotonous with respect to the choice of C and I: $C \preceq C' \Rightarrow \delta^C \preceq \delta^{C'}$, where δ^C is the dilation defined by Eq. (22) using the bipolar conjunction or t-norm C, and $I \preceq I' \Rightarrow \varepsilon^I \preceq \varepsilon^{I'}$, where ε^I is the erosion defined by Eq. (21) using the bipolar implication I.*

More properties on the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are provided in Section 6.3.

4. Pareto (marginal) partial ordering

In this section, we detail the case of Pareto ordering, in order to illustrate the general definitions and results of Section 3. This summarizes our previous results in [15,16,19], and includes additional properties concerning ordering when using different bipolar conjunctions and implications, as well as a discussion on this ordering. Using this ordering, the positive and negative components of information are handled in a symmetric way. This is further discussed at the end of this section.

4.1. Complete lattice derived from Pareto ordering and connectives

The marginal partial ordering on $\mathcal{L}$, or Pareto ordering (by reversing the scale of negative information) is defined as:

$$(a_1, b_1) \preceq_p (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \geq b_2. \tag{27}$$

This ordering, often used in economics and social choice, has also been used for bipolar information [51], intuitionistic fuzzy sets e.g. in [35], or interval-valued fuzzy sets [73].

For this partial ordering, $(\mathcal{L}, \preceq_p)$ is a complete lattice. The greatest element is $(1, 0)$ and the smallest element is $(0, 1)$. The supremum and infimum are respectively defined as:

$$(a_1, b_1) \vee_p (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2)), \tag{28}$$

$$(a_1, b_1) \wedge_p (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2)). \tag{29}$$

The partial order $\preceq_p$ induces a partial order on the set of bipolar fuzzy sets:

Definition 6. A Pareto ordering on $\mathcal{B}$ is defined as: $\forall(\mu_1, \nu_1) \in \mathcal{B}, \forall(\mu_2, \nu_2) \in \mathcal{B}$,

$$(\mu_1, \nu_1) \preceq_p (\mu_2, \nu_2) \text{ iff } \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x) \text{ and } \nu_1(x) \geq \nu_2(x). \tag{30}$$

Note that this corresponds formally to the inclusion on intuitionistic fuzzy sets [6] (again the semantics are different).

$(\mathcal{B}, \preceq_p)$ is a complete lattice. The supremum and the infimum of any family of bipolar fuzzy sets (μ_i, ν_i), $i \in I$, where the index set I can be finite or not, are given by:

$$\forall x \in \mathcal{S}, \bigvee_{p\ i\in I}(\mu_i, \nu_i)(x) = \left(\sup_{i\in I} \mu_i(x), \inf_{i\in I} \nu_i(x)\right),$$

$$\forall x \in \mathcal{S}, \bigwedge_{p\ i\in I}(\mu_i, \nu_i)(x) = \left(\inf_{i\in I} \mu_i(x), \sup_{i\in I} \nu_i(x)\right),$$

The greatest element is the pair of functions $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ constantly equal $1_{\mathcal{L}}$, and the smallest element is the pair of functions (μ_0, ν_0) constantly equal to $0_{\mathcal{L}}$.

Let us now mention a few connectives. In Definition 2, the monotony properties have now to be intended according to the Pareto ordering.

An example of negation, which will be used in the following, is the standard negation, defined by $N((a, b)) = (b, a)$.

Two types of t-norms and t-conorms are considered in [42] (actually in the intuitionistic case) and will be considered here as well in the bipolar case (more details on different classes of operators can be found in [41]):

(1) Operators called t-representable bipolar t-norms and t-conorms, which can be expressed using usual t-norms t and t-conorms T:

$$C((a_1, b_1), (a_2, b_2)) = (t(a_1, a_2), T(b_1, b_2)), \quad (31)$$

$$D((a_1, b_1), (a_2, b_2)) = (T(a_1, a_2), t(b_1, b_2)). \quad (32)$$

A typical example is obtained for $t = \min$ and $T = \max$. Although t and T are usually chosen as dual operators, other choices are possible, as discussed e.g. in [72] for adjunction properties. Distributivity properties of implications over t-norms are further investigated in [8].

(2) Bipolar Lukasiewicz operators, which are not t-representable:

$$C_W((a_1, b_1), (a_2, b_2)) = (\max(0, a_1 + a_2 - 1), \min(1, b_1 + 1 - a_2, b_2 + 1 - a_1)), \quad (33)$$

$$D_W((a_1, b_1), (a_2, b_2)) = (\min(1, a_1 + 1 - b_2, a_2 + 1 - b_1), \max(0, b_1 + b_2 - 1)). \quad (34)$$

In these equations, the positive part of C_W is the usual Lukasiewicz t-norm of a_1 and a_2 (i.e., the positive parts of the input bipolar values). The negative part of D_W is the usual Lukasiewicz t-norm of the negative parts (b_1 and b_2) of the input values. Hence these operators are called pessimistic t-norm and optimistic t-conorm, respectively, in [41] (C_W and D_W are actually two examples of such operators).

The two types of implication introduced in Section 2 can be used here as well, and were also considered in [34,42]. The two types of implication coincide for the Lukasiewicz operators C_W and D_W [35].

Proposition 17. *Let us denote by C_{min} (respectively D_{max}) the t-representable bipolar conjunction (respectively disjunction) built from the minimum and maximum, and C_{prod} (respectively D_{sum}) the one built from the product and algebraic sum. We have the following ordering between conjunctions:* $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,

$$C_W((a_1, b_1), (a_2, b_2)) \preceq_p C_{prod}((a_1, b_1), (a_2, b_2)) \preceq_p C_{min}((a_1, b_1), (a_2, b_2)), \quad (35)$$

and for disjunctions:

$$D_{max}((a_1, b_1), (a_2, b_2)) \preceq_p D_{sum}((a_1, b_1), (a_2, b_2)) \preceq_p D_W((a_1, b_1), (a_2, b_2)). \quad (36)$$

Morphological operators derived from these connectives will inherit these properties.

4.2. Algebraic and morphological erosions and dilations

Since the Pareto ordering is an example leading to complete lattices, algebraic dilations and erosions can be defined as in Section 3.1.

Next, introducing structuring elements, morphological erosions and dilations are defined as in Eqs. (21) and (22). Details can be found in [15,16,19].

It is easy to show that the bipolar Lukasiewicz operators C_W and D_W are adjoint, according to Eq. (2). Therefore, if these Lukasiewicz operators (up to a bijection) are used, then all algebraic properties detailed in Section 3.1 hold. Moreover, it has been shown that the adjoint operators which are moreover continuous, Archimedian, nilpotent and satisfy some boundary conditions are all derived from the Lukasiewicz operator C_W and D_W, using a continuous bijective permutation on [0, 1] [42] (Theorem 8.10). Hence having dilations and erosions that are both dual and adjoint can be achieved for this class of operators. This completes the result of Theorem 1 in the particular case of the Pareto ordering, as also mentioned for this case in [16,19,79,97].

4.3. Interpretation

In order to interpret the expression of morphological erosion, let us first consider the implication I defined from a t-representable bipolar t-conorm D, i.e., $\forall((a,b),(a',b')) \in \mathcal{L}^2, I((a,b),(a',b')) = D((b,a),(a',b'))$, when using the standard negation, and D is defined as in Eq. (32). Then the erosion is expressed as:

$$\begin{aligned}\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \bigwedge\nolimits_{p\; y\in\mathcal{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y)))\\ &= \bigwedge\nolimits_{p\; y\in\mathcal{S}} (T((\nu_B(y-x),\mu(y)),t(\mu_B(y-x),\nu(y)))\\ &= \left(\inf_{y\in\mathcal{S}} T((\nu_B(y-x),\mu(y)),\sup_{y\in\mathcal{S}} t(\mu_B(y-x),\nu(y)))\right).\end{aligned} \tag{37}$$

The second line is derived from the fact that D is supposed here to be a t-representable bipolar t-conorm, defined from a t-norm t and a t-conorm T. The third line is derived from the definition of the infimum in $\mathcal{L}$ and in $\mathcal{B}$ for $\preceq_p$. This resulting bipolar fuzzy set has a membership function which is exactly the fuzzy erosion of μ by the fuzzy structuring element $1-\nu_B$, according to the original definitions in the fuzzy case [22]. The non-membership function is exactly the dilation of the fuzzy set ν by the fuzzy structuring element μ_B.

Let us consider the dilation, defined from a t-representable t-norm C (Eq. (31)). Using the standard negation, it is expressed as:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \left(\sup_{y\in\mathcal{S}} t(\mu_B(x-y),\mu(y)),\inf_{y\in\mathcal{S}} T((\nu_B(x-y),\nu(y)))\right). \tag{38}$$

The first term (membership function) is exactly the fuzzy dilation of μ by μ_B, while the second one (non-membership function) is the fuzzy erosion of ν by $1-\nu_B$, according to the original definitions in the fuzzy case [22].

This observation has a nice interpretation, which well fits with intuition. Let (μ,ν) represent a spatial bipolar fuzzy set, where μ is a positive information for the location of an object for instance, and ν a negative information for this location. A bipolar structuring element can represent additional imprecision on the location, or additional possible locations. Dilating (μ,ν) by this bipolar structuring element amounts to dilating μ by μ_B, i.e., the positive region is extended by an amount represented by the positive information encoded in the structuring element. On the contrary, the negative information is eroded by the complement of the negative information encoded in the structuring element. This corresponds well to what would be intuitively expected in such situations. A similar interpretation can be provided for the bipolar fuzzy erosion. Examples are provided in the next subsection.

Let us now consider the implication derived from the Lukasiewicz bipolar operators C_W and D_W (Eqs. (33) and (34)). The erosion and dilation expressions become:

$$\begin{aligned}&\forall x\in\mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x)\\ &= \bigwedge\nolimits_{p\; y\in\mathcal{S}} (\min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)),\max(0,\nu(y)+\mu_B(y-x)-1))\\ &= \left(\inf_{y\in\mathcal{S}} \min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)),\sup_{y\in\mathcal{S}} \max(0,\nu(y)+\mu_B(y-x)-1)\right),\\ &\forall x\in\mathcal{S},\ \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x)\\ &= \left(\sup_{y\in\mathcal{S}} \max(0,\mu(y)+\mu_B(x-y)-1),\inf_{y\in\mathcal{S}} \min(1,\nu(y)+1-\mu_B(x-y),\nu_B(x-y)+1-\mu(y))\right).\end{aligned}$$

The negative part of the erosion is exactly the fuzzy dilation of ν (negative part of the input bipolar fuzzy set) with the structuring element μ_B (positive part of the bipolar fuzzy structuring element), using the usual Lukasiewicz t-norm. Similarly, the positive part of the dilation is the fuzzy dilation of μ (positive part of the input) by μ_B (positive part of the bipolar fuzzy structuring element), using the usual Lukasiewicz t-norm. Hence for both operators, the "dilation" part (i.e., negative part for the erosion and positive part for the dilation) has always a direct interpretation and is the same as the one obtained using t-representable operators, for t being the usual Lukasiewicz t-norm.

In the case where the structuring element is non bipolar (i.e., $\forall x\in\mathcal{S}, \nu_B(x) = 1-\mu_B(x)$, or $\nu_B = 0$), then the "erosion" part has also a direct interpretation: the positive part of the erosion is the fuzzy erosion of μ by μ_B for the usual Lukasiewicz t-conorm; the negative part of the dilation is the erosion of ν by μ_B for the usual Lukasiewicz t-conorm.

It follows from Propositions 16 and 17 that the some erosions and dilations can be ordered according to the used connectives.

Proposition 18. *Let us denote by δ^{min}, δ^{prod} and δ^{W} the dilations built from C_{min}, C_{prod} and C_W, respectively. We have the following ordering:* $\forall((\mu_B, \nu_B), (\mu, \nu)) \in \mathcal{B}^2$,

$$\delta^{W}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \delta^{prod}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \delta^{min}_{(\mu_B,\nu_B)}(\mu, \nu). \quad (39)$$

Let us denote by ε^{max}, ε^{sum} and ε^{W} the erosions built from the implications derived from D_{max}, D_{sum} and D_W, respectively. We have the following ordering: $\forall((\mu_B, \nu_B), (\mu, \nu)) \in \mathcal{B}^2$,

$$\varepsilon^{max}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \varepsilon^{sum}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \varepsilon^{W}_{(\mu_B,\nu_B)}(\mu, \nu). \quad (40)$$

This means that operations built from min and max have a stronger effect on the initial bipolar fuzzy set.

Let us finally comment on the practical use of these operators, where discretization may induce some approximations. This question has already been addressed in the case of interval-valued fuzzy sets in [74]. The discretization of the space $\mathcal{S}$ does not induce any particular problem. As for the values of μ and ν, the discretization of $[0, 1]$ may induce some small errors depending on the choice of C and I. Let us assume that the values are regularly discretized (as is usually the case), in the form $\frac{k}{n}$ where k and n are integer values, with n defining the granularity of the discretization and $0 \leq k \leq n$. The negation, minimum, maximum and Lukasiewicz operators provide exact results, and hence C_{min}, C_W, D_{max}, D_W, δ^{min}, δ^{W}, ε^{max}, ε^{W}. However the product and algebraic sum (and thus C_{prod}, D_{sum}, δ^{prod}, ε^{sum}) need some approximation. For a quantification on 6 to 10 bits ($n = 2^6 - 1$ to $n = 2^{10} - 1$), we have tested that the maximal error on the product does not exceed the quantification step $\frac{1}{n}$. Therefore the approximation errors can be considered as low enough to be neglected in the applications.

4.4. Illustrative example in the spatial domain

When dealing with spatial information, in image processing or for spatial reasoning applications, bipolarity may be an important feature of the information to be processed. It can be simply related to grey levels and imprecision attached to them, but it can also represent more complex and structural types of information for higher level image interpretation. For instance, when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indetermination. This corresponds to the idea that the union of positive and negative information does not cover all the space. Similar considerations can be provided for other pairs of "opposite" relations, such as "close to" and "far from" for instance. Note that considering "opposite" relations is but one example. Other examples could be provided, where relations could be not opposite, or even of different nature. For instance: we have some positive information for an object being above a reference object (directional relation), and some negative information for the object not being in some region of space (topological relation).

An example is illustrated in Fig. 1. It shows an object at some position in the space (the rectangle in this figure). For visualization purposes, in all illustrations, a representation using grey levels is adopted for encoding $\mu(x)$ and $\nu(x)$ (0 = black, 1 = white). Two images are shown, one for positive information and one for negative information. Let us assume that some information about the position of another object is provided: it is to the left of the rectangle and not to the right. The region "to the left of the rectangle" is computed using a fuzzy dilation with a directional fuzzy structuring element providing the semantics of "to the left" [14], thus defining the positive information. The region "to the right of the rectangle" defines the negative information and is computed in a similar way. The membership functions μ_L and μ_R represent respectively the positive and negative parts of the bipolar fuzzy set. They are not the complement of each other, and we have: $\forall x$, $\mu_L(x) + \mu_R(x) \leq 1$. Here we assume that this consistency constraint hold. A discussion on how to achieve it can be found in [18].

To our knowledge, bipolarity has not been much exploited in the spatial domain. A few works deal with image thresholding, filtering, comparison or edge detection, based on intuitionistic fuzzy sets derived from image intensity and entropy or divergence criteria [12,26,27,29,31,36,101]. Spatial representations of interval-valued fuzzy sets have also been proposed in [32], as a kind of fuzzy egg-yolk,[2] for evaluating classification errors based on ground-truth, or in [69,70] with preliminary extensions of RCC to these representations. Interval-valued fuzzy sets have been used in [75,79,97] to represent uncertainty about grey levels, and morphological operators have been used in this context. Another recent application of

[2] A formalism for qualitative spatial reasoning, based on region connection calculus, where a vague region A is represented with a set of points that definitely belong to the vague region (the yolk), and a set (the white of the "egg") whose complement contains the points that definitely do not [58]. The two sets can be fuzzy in the fuzzy extension of this formalism.

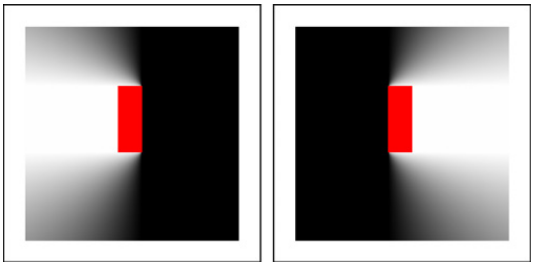

Fig. 1. Region to the left of the rectangle (positive information, μ_L) and region to the right of the rectangle (negative information, μ_R). The membership degrees vary from 0 (black) to 1 (white).

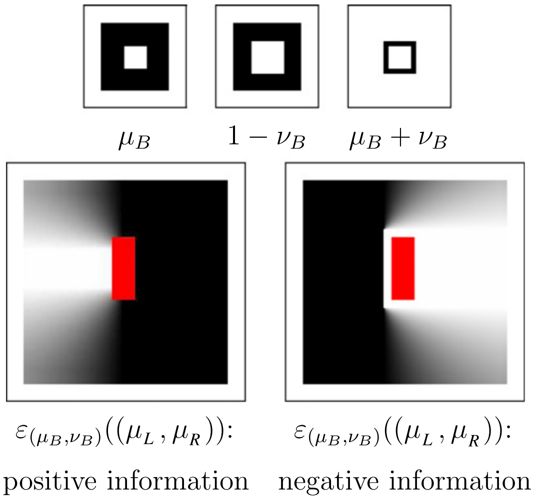

Fig. 2. Illustration of a bipolar fuzzy erosion on the example shown in Fig. 1. The results, displayed on the second line, show the reduction of the positive part and the extension of the negative part. The image has a size of 256 × 256 pixels, while the structuring element has a support of 20 × 20 pixels.

interval-valued fuzzy sets was proposed for stereo matching taking intro account the uncertainty at each point in [52]. An interesting approach to derive interval-valued fuzzy sets from conjunction and disjunction of grey levels computed in a spatial neighborhood of each point was proposed in [27], and then further used for edge detection. This approach has clear links with (classical) mathematical morphology since the interval bounds correspond to erosion and dilation performed with a crisp structuring element, while their difference, used for deriving edges, corresponds to the morphological gradient. But, apart these few works, there are still very few tools for manipulating spatial information using both its bipolarity and imprecision components, and they mostly handle only low level information. A simple example for spatial reasoning can be found in our previous work [16]. Note that here we do not make any restrictive assumption on the origin of the bipolar information to handle. We may derive positive information from the colors or shape of the regions of the image, or from relational information with respect to other objects found in the image (as illustrated in this section), and negative information from other colors, other reference objects, or any other type of information (including coming from another image or another source, different from the image itself). In particular, we do not restrict to the cases where there is a direct link between grey levels or colors and μ and ν. This differs from other applications of intuitionistic or interval-valued fuzzy sets in image processing (e.g., [27,31,36,75,79,97,101]).

Let us now illustrate the proposed morphological operations on the simple example shown in Fig. 1. Let us assume that an additional information, given as a bipolar structuring element, allows us to reduce the positive part and to extend the negative part of the bipolar fuzzy region. This can be formally expressed as a bipolar fuzzy erosion, applied to the bipolar fuzzy set (μ_L, μ_R), using this structuring element. Fig. 2 illustrates the result. It can be observed that the region corresponding to the positive information has actually been reduced (via a fuzzy erosion), while the region corresponding to the negative part has been extended (via a fuzzy dilation). In all examples, ε^W and δ^W have been used, but similar effects are obtained with operations defined from other connectives.

4.5. Discussion on Pareto ordering

The Pareto ordering implies that positive information and negative information play symmetrical roles and are handled similarly. This ordering is often used, in various domains, and it may be sometimes an advantage to consider this symmetry. Then the proposed definitions apply and dilations and erosions have nice interpretations, well fitting the intuition in such cases.

However, based on the discussion about semantics in Section 2.4, this might not always be appropriate, since we may want to process positive and negative information in different ways when dealing with asymmetric bipolarity, in particular when the two types of information are issued from different sources or have different semantics. For instance if the positive information represents preferences and the negative information rules or constraints, then it may be interesting (or mandatory) to give more priority to the constraints (or in the contrary to the positive information) [9,10,49,65,82]. The partial ordering should then be replaced by another one, accounting for these priorities. This will be addressed in Section 5.

Another debatable point is the standard negation. While it is consistent with a symmetric interpretation of bipolarity, it might not be for asymmetric bipolarity. If positive and negative parts of the information are of different nature, it may indeed not make sense to just exchange them. Note that a complete characterization of intuitionistic negations has been developed in [42].

As discussed in [44], Pareto partial ordering may be too partial, in the sense that incomparability occurs very often, even in situations where comparison could intuitively be done, as illustrated by examples in [44]. Another drawback discussed in this work is that it does not take into account the scale of the two components and lacks focalization. For instance the comparison between (a, b) and (a', b') depends on the order between a and a', and on the order between b and b', but not on the comparison between by how much a is larger or smaller than a' and by how much b is larger or smaller than b'. Different comparison rules have been proposed in [44] to overcome this limitation.

Moreover, in the particular context of mathematical morphology, this ordering has an additional drawback: the value at a point in the resulting dilation or erosion is generally expected to be one of the values of neighborhood points (defined by the structuring element), but this is in general not the case when using Pareto ordering. This point has already raised discussions in the mathematical morphology community, in particular when dealing with vector-valued images, such as color images (see e.g., [3,5,99]). It has been shown that non vector-preserving orderings may lead to counter-intuitive results. For instance introducing new colors (e.g. in a morphological filtering process), that do not belong to any of the image objects, may prevent their correct recognition, if this recognition is based on colors and cannot handle unexpected colors. Wrong recognition may also occur when an operation modifies an object color by incidentally introducing colors that may be characteristic of another object.

5. Bipolar fuzzy mathematical morphology based on lexicographic ordering

In this section, as a second example, we introduce priorities between the two types of information, thus handling asymmetry, based on a lexicographic ordering which induces another way of modeling mathematical morphology. It also guarantees that the resulting bipolar value at a point is one of the values of neighborhood points. Thus, this addresses some of the issues mentioned in Section 4.5. The lexicographic ordering (also called dictionary ordering) is denoted by $\preceq_L$. It is additionally a total order on $\mathcal{L}$. On the induced lattice on $\mathcal{B}$, we define algebraic dilations and erosions. We also propose connectives that are adapted to this ordering, and then derive morphological dilations and erosions, as in [19], where more details can be found. A brief discussion on this ordering is then proposed.

5.1. Lexicographic ordering and associated lattice

Definition 7. The lexicographic relation $\preceq_L$ on $\mathcal{L}$, giving priority to negative information, is defined as:

$$(a, b) \preceq_L (a', b') \Leftrightarrow b > b' \text{ or } (b = b' \text{ and } a \leq a'). \tag{41}$$

The relation $\preceq_L$ defines a total ordering on $\mathcal{L}$ and $(\mathcal{L}, \preceq_L)$ is a complete lattice (more specifically a chain in this case, as the ordinal product of two chains [13,37]). The smallest element is $(0, 1)$ and the largest element is $(1, 0)$.

A lexicographic ordering giving priority to the positive information can be defined in a similar way. All what follows applies in both cases, and we only detail the one of Definition 7 in this paper.

Fig. 3 illustrates the difference between $\preceq_p$ and $\preceq_L$.

This ordering induces a partial ordering on $\mathcal{B}$ (the same notation is used):

Definition 8. The lexicographic relation on $\mathcal{B}$ is defined by:

$$(\mu, \nu) \preceq_L (\mu', \nu') \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x), \nu(x)) \preceq_L (\mu'(x), \nu'(x)). \tag{42}$$

This definition means that a bipolar fuzzy set is considered as smaller than another one if its negative part is larger, or if the two negative parts are equal and the positive part is smaller. This strongly expresses the priority given to the negative information, since only the negative parts are considered as soon as they differ.

The relation $\preceq_L$ (Definition 8) defines a partial ordering, called lexicographic ordering, on $\mathcal{B}$ and $(\mathcal{B}, \preceq_L)$ is a complete lattice. The smallest element is (μ_0, ν_0) (defined by $\forall x \in \mathcal{S}$, $\mu_0(x) = 0$, $\nu_0(x) = 1$), and the largest element is $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ (defined by $\forall x \in \mathcal{S}$, $\mu_{\mathbb{I}}(x) = 1$, $\nu_{\mathbb{I}}(x) = 0$). Infimum and supremum for $\preceq_L$ are expressed, for any two elements (a, b) and (a', b') of $\mathcal{L}$, as:

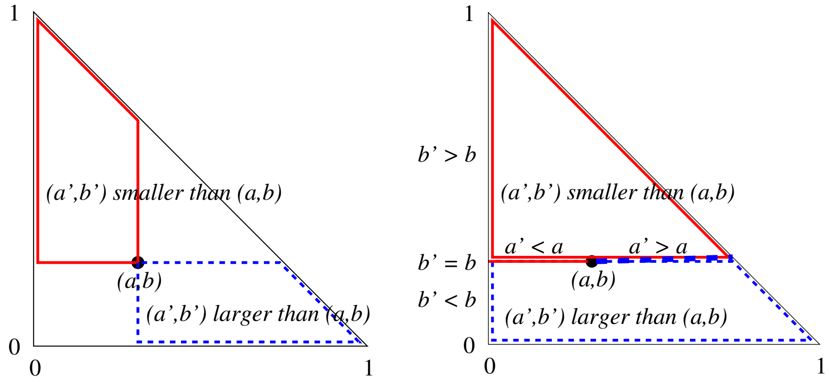

Fig. 3. Comparison, in $\mathcal{L}$, between the partial ordering $\preceq_p$ (left) and the total ordering $\preceq_L$ (right). Plain (respectively dashed) lines indicate the regions of $\mathcal{L}$ in which points (a', b') are smaller (respectively larger) than point (a, b).

$$\min_{\preceq_L}((a,b),(a',b')) = \begin{cases} (a,b) & \text{if } b > b' \\ (a',b') & \text{if } b < b' \\ (\min(a,a'),b) & \text{if } b = b' \end{cases} \tag{43}$$

$$\max_{\preceq_L}((a,b),(a',b')) = \begin{cases} (a,b) & \text{if } b < b' \\ (a',b') & \text{if } b > b' \\ (\max(a,a'),b) & \text{if } b = b' \end{cases} \tag{44}$$

Infimum and supremum for any family of elements of $\mathcal{L}$ or $\mathcal{B}$ are derived in a straightforward way, and are denoted by $\bigwedge_{\preceq_L}$ and $\bigvee_{\preceq_L}$. They can be computed using fast sorting algorithms.

Let us note that, in all cases, the lexicographic minimum (or maximum) provides a result which is one of the input bipolar values.

5.2. Connectives

Bipolar connectives are defined as in Section 2.3, considering monotonicity with respect to $\preceq_L$.

With respect to the Pareto ordering $\preceq_p$, the standard negation $N((a,b)) = (b,a)$ is decreasing. However it is not for the lexicographic ordering $\preceq_L$ and is hence not a negation. Therefore, we propose a new definition of negation, illustrated in Fig. 4 [19].

Definition 9. The natural negation $n_{\preceq_L}$ associated with the lexicographic ordering is defined as an operator that reverses the ordering of the elements of $\mathcal{L}$. In the discrete case, it is defined based on a one-to-one correspondence between the scale and the reversed scale (i.e., the negation of the largest element is the smallest one, the negation of the penultimate is the second one, etc.).

This definition of $n_{\preceq_L}$ is actually a negation (involutive and decreasing). This result is derived from the fact that $\preceq_L$ is a total ordering on $\mathcal{L}$.

In the continuous case, there are several possibilities for defining explicitly such a negation reversing the scale. Here we restrict to the discrete case and propose an explicit computation in that case. Although an explicit expression is not straightforward to obtain, from an algorithmical point of view, the computation of the negation is simple when the levels between 0 and 1 are discrete, i.e., take only a finite number of values (which is generally the case in practical applications). We tabulate the ranks of (a_i, b_j), for i and j varying from 0 to N if the interval $[0, 1]$ is discretized on $N + 1$ levels $\left(\text{for instance } a_i = \frac{i}{N},\ b_j = \frac{j}{N}\right)$. The rank of $\left(\frac{i}{N}, \frac{j}{N}\right)$ is $r_{ij} = \frac{(N-j+1)(N-j)}{2} + i$ and the rank of $n_{\preceq_L}\left(\frac{i}{N}, \frac{j}{N}\right)$ is equal to $\frac{(N+1)(N+2)}{2} - 1 - r_{ij}$. The continuous (infinite) case would be however more complicated and is not considered here. From a geometrical point of view, the negation of a point (a, b) is the point $n_{\preceq_L}(a, b)$ such that the number of points in the triangle comprising the points smaller than (a, b) (see Fig. 3) is equal to the number of points in the trapeze formed by the points that are larger than $n_{\preceq_L}(a, b)$.

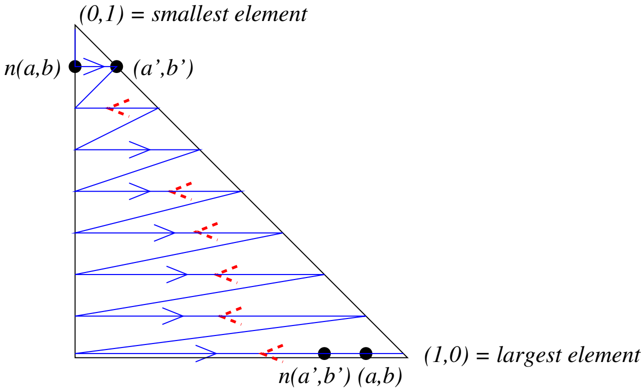

Fig. 4. Natural negation for the lexicographic ordering. Plain arrows indicate the ordering from the smallest to the largest element of $\mathcal{L}$ and the dashed arrows indicate the reverse order. Two examples of points (a, b) and (a', b') and their negations $n_{\preceq_L}(a, b)$ and $n_{\preceq_L}(a', b')$ are shown.

Proposition 19. *The minimum* $\min_{\preceq_L}$ *and maximum* $\max_{\preceq_L}$ *associated with the lexicographic ordering are bipolar t-norms and t-conorms on the lattice* $(\mathcal{L}, \preceq_L)$.[3] *Moreover they are idempotent and mutually distributive,* $\min_{\preceq_L}$ *is the largest t-norm and* $\max_{\preceq_L}$ *the smallest t-conorm (according to* $\preceq_L$*). They are also dual with respect to the negation* $n_{\preceq_L}$.

From Proposition 1, bipolar implications are derived, and take here the following particular forms:

- $\forall (a, b) \in \mathcal{L}, \forall (a', b') \in \mathcal{L},$
$$I_N((a, b), (a', b')) = \max_{\preceq_L}(n_{\preceq_L}(a, b), (a', b')) \tag{45}$$
- $\forall (a, b) \in \mathcal{L}, \forall (a', b') \in \mathcal{L},$
$$I_R((a, b), (a', b')) = \bigvee_{\preceq_L}\{(\alpha, \beta) \in \mathcal{L} \mid \min_{\preceq_L}((a, b), (\alpha, \beta)) \preceq_L (a', b')\} \tag{46}$$
(adjoint implication of the t-norm $\min_{\preceq_L}$). A closed-form expression is as follows:
$$I_R((a, b), (a', b')) = \begin{cases} (1, 0) & \text{if } b > b' \\ (a', b') & \text{if } b < b' \\ (1, 0) & \text{if } b = b' \text{ and } a \leq a' \\ (a', b') & \text{if } b = b' \text{ and } a > a' \end{cases} \tag{47}$$
or, equivalently:
$$I_R((a, b), (a', b')) = \begin{cases} (1, 0) = 1_{\mathcal{L}} & \text{if } (a, b) \preceq_L (a', b') \\ (a', b') & \text{if } (a', b') \prec_L (a, b) \end{cases} \tag{48}$$

5.3. Algebraic and morphological dilations and erosions on the lattice $(\mathcal{B}, \preceq_L)$

Since $(\mathcal{B}, \preceq_L)$ is a complete lattice, algebraic dilations and erosions can be defined as in Section 3.1, as operators that commute with $\bigvee_{\preceq_L}$ and $\bigwedge_{\preceq_L}$, respectively. Similarly, the adjunction is defined with respect to $\preceq_L$. The properties of these operators and their compositions (in particular closing and opening) are directly derived from the properties of complete lattices and are the same as those described in Section 3.1 for the general case.

Let us now consider the case where $\mathcal{S}$ is an affine space, on which translations are defined. Again, we define a degree of intersection as the supremum of a bipolar t-norm C and a degree of inclusion as the infimum of a bipolar implication I, where the bipolar connectives are defined according to $\preceq_L$. Let (μ_B, ν_B) be a bipolar structuring element (in $\mathcal{B}$). The dilation and erosion of any element (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) are then expressed as:

$$\forall x \in \mathcal{S}, \;\; \delta_{(\mu_B, \nu_B)}((\mu, \nu))(x) = \bigvee_{\preceq_L y \in \mathcal{S}} C((\mu_B(x - y), \nu_B(x - y)), (\mu(y), \nu(y))). \tag{49}$$
$$\forall x \in \mathcal{S}, \;\; \varepsilon_{(\mu_B, \nu_B)}((\mu, \nu))(x) = \bigwedge_{\preceq_L y \in \mathcal{S}} I((\mu_B(y - x), \nu_B(y - x)), (\mu(y), \nu(y))). \tag{50}$$

[3] Note that $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are not increasing with respect to $\preceq_p$ and are therefore not t-norms and t-conorms on $(\mathcal{L}, \preceq_p)$.

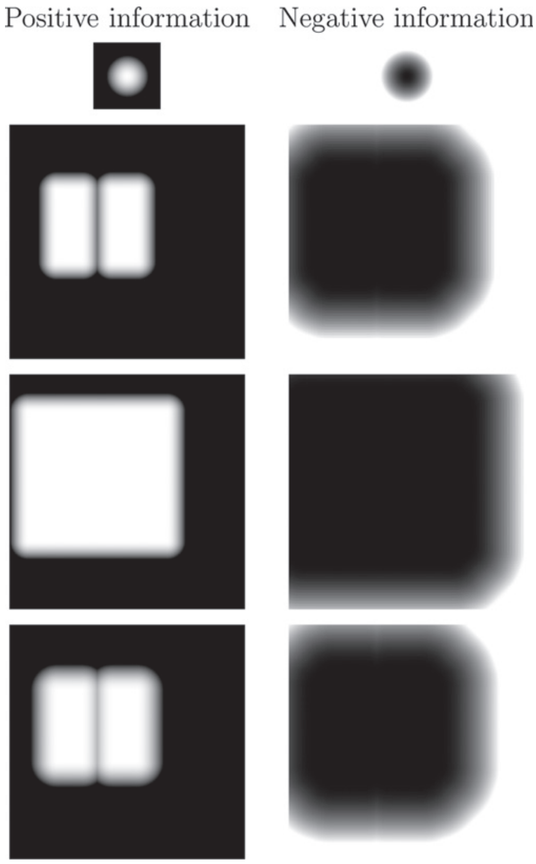

Fig. 5. From top to bottom: bipolar fuzzy structuring element (the support of μ_B has a size of 38 × 38 pixels, and the one of $1 - \nu_B$ 48 × 48), original bipolar fuzzy set, dilation using the lexicographic minimum, dilation using Pareto ordering and the associated minimum as conjunction, for the sake of comparison.

It should be noted that, as in Section 3.2, a bipolar t-norm (i.e., a stronger operator than a general bipolar conjunction) is involved in the proposed definition (Eq. (49)), so as to guarantee good properties. For the erosion (Eq. (50)), both types of implications I_N and I_R can be used, with somewhat different properties. The dilation defined from $\min_{\preceq_L}$ and the erosion defined from I_N (for $\max_{\preceq_L}$ and the negation $n_{\preceq_L}$) are dual with respect to the negation $n_{\preceq_L}$: $\delta_{(\mu_B,\nu_B)}(n_{\preceq_L}(\mu, \nu)) = n_{\preceq_L}(\varepsilon_{(\mu_B,\nu_B)}(\mu, \nu))$. The dilation defined from $\min_{\preceq_L}$ and the erosion defined from I_R (residual implication of $\min_{\preceq_L}$) are adjoint. It follows that all general algebraic properties described in Section 3 hold in that case.

Note that the two properties of adjunction and of duality are not simultaneously satisfied for these operators (since the dual operator of $\min_{\preceq_L}$ is $\max_{\preceq_L}$ but it is not its adjoint). It would be interesting to prove the existence and then build operators equivalent to Lukasiewicz ones, for $\preceq_L$, so as to derive results similar to those in the fuzzy case (see e.g., [17,22]) and in the bipolar fuzzy case for the Pareto ordering (see Section 4). It follows that the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are true opening and closing if $\min_{\preceq_L}$ and I_R are used (because of the adjunction property), while they are not if $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are used ($\delta\varepsilon$ and $\varepsilon\delta$ are not idempotent in this case).

5.4. Illustrative example

An example is illustrated in Fig. 5, using the lexicographic minimum $\min_{\preceq_L}$ as a t-norm. As expected, the dilation extends the positive parts and reduces the negative parts. The priority given to the negative parts and the fact that $\min_{\preceq_L}$ always provides one of the input values (which is not the case of the Pareto ordering) induces a stronger effect of the transformation when using the lexicographic ordering (the Pareto minimum has the same negative part than $\min_{\preceq_L}$ and a smaller positive part). For instance, at a point where the structuring element covers a part with non-zero positive information and zero negative information, the dilation with Pareto ordering and Pareto minimum may lead to $0_{\mathcal{L}}$ at this point, while the dilation with lexicographic minimum may lead to $(a, 0)$ with $a \neq 0$. This is typically what happens in the bright regions of the positive part of the lexicographic dilation, which remain black in the Pareto dilation.

5.5. Discussion on lexicographic ordering

As mentioned in Section 4.5, the lexicographic ordering answers some of the questions raised by Pareto ordering. It preserves the asymmetry of the information, and the derived morphological operators are vector-preserving, which is a desirable feature for applications in image processing.

However, it also raises a number of issues. Although the priority can be given either to the negative information (as detailed here) or to the positive one depending on the problem and on the application, this priority can be sometimes too strong, leading to situations where the other part of the information is seldom taken into account. This was addressed in the color image processing community by allowing more frequent comparisons of the other component values (for instance by using a rougher quantization of the component to which priority is given).

Another issue in image processing is the interpretation and visual representation of the negation. While it is straightforward and natural in $\mathcal{L}$, it is not so in the spatial domain, and it is more difficult to represent this negation using grey levels than the standard negation related to Pareto ordering. It would be useful to find a better visual representation in this particular context, which may depend on what from the image is actually represented as positive and negative information.

Some future work should also investigate other connectives associated with this ordering.

6. Derived operators

Once the two basic morphological operators, erosion and dilation, have been defined on bipolar fuzzy sets, other operators can be derived in a quite straightforward way. We provide a few examples in this section, coming back to the general setting with any partial ordering if not otherwise specified.

6.1. Morphological gradient

A direct application of erosion and dilation is the morphological gradient, which extracts boundaries of objects by computing the difference between dilation and erosion [91]. We propose here an extension to the bipolar fuzzy case.

Definition 10. Let (μ, ν) a bipolar fuzzy set. We denote its dilation by a bipolar fuzzy structuring element by (δ^+, δ^-) and its erosion by $(\varepsilon^+, \varepsilon^-)$. We define the bipolar fuzzy gradient as:

$$\nabla(\mu, \nu) = \bigwedge(N(\varepsilon^+, \varepsilon^-), (\delta^+, \delta^-)) \tag{51}$$

which is the set difference, expressed as the conjunction between (δ^+, δ^-) and the negation of $(\varepsilon^+, \varepsilon^-)$.

For instance, in the case of Pareto ordering and standard negation, the gradient is expressed as $\nabla(\mu, \nu) = (\min(\delta^+, \varepsilon^-), \max(\delta^-, \varepsilon^+))$. Another suggestion in [79] in the interval-valued fuzzy sets setting is to take the interval difference between $[\delta^+, 1 - \delta^-]$ and $[\varepsilon^+, 1 - \varepsilon^-]$, i.e., $[\delta^+ - 1 + \varepsilon^-, \max(\delta^+ - \varepsilon^+, \varepsilon^- - \delta^-)]$.

Proposition 20. *The bipolar fuzzy gradient has the following properties:*

(1) *Definition 10 defines a bipolar fuzzy set.*
(2) *If the dilation and erosion are defined in the case of Pareto ordering and using t-representable bipolar t-norms and t-conorms, we have:*

$$\nabla(\mu, \nu) = (\min(\delta_{\mu_B}(\mu), \delta_{\mu_B}(\nu)), \max(\varepsilon_{1-\nu_B}(\nu), \varepsilon_{1-\nu_B}(\mu))). \tag{52}$$

Moreover, if (μ, ν) is not bipolar (i.e., $\nu = 1 - \mu$, or $\nu = 0$), then the positive part of the gradient is equal to $\min(\delta_{\mu_B}(\mu), 1 - \varepsilon_{\mu_B}(\mu))$, which is exactly the morphological gradient in the fuzzy case.

These results follow directly from the expressions of bipolar dilations and erosions. An illustration is displayed in Fig. 6. It illustrates both the imprecision (through the fuzziness of the gradient) and the indetermination (through the indetermination between the positive and the negative parts). The object is here somewhat complex, and exhibits two different parts, that can be considered as two connected components to some degree. The positive part of the gradient provides a good account of the boundaries of the union of the two components, which amounts to considering that the region between the two components, which has lower membership degrees, actually belongs to the object. The positive part has the expected interpretation as a surely possible position and spatial extension of the contours. The negative part shows the level of indetermination in the gradient: the gradient could be larger as well, and it could also include the region between the two components.

The choice of the bipolar t-norms and t-conorms used for computing the dilation and the erosion has an influence on the result, with more or less effect, resulting from Proposition 16. In the case of Pareto ordering, finer results using C_W and D_W will be obtained than when using C_{min} and D_{max}, or C_{prod} and D_{sum} (see Proposition 17).

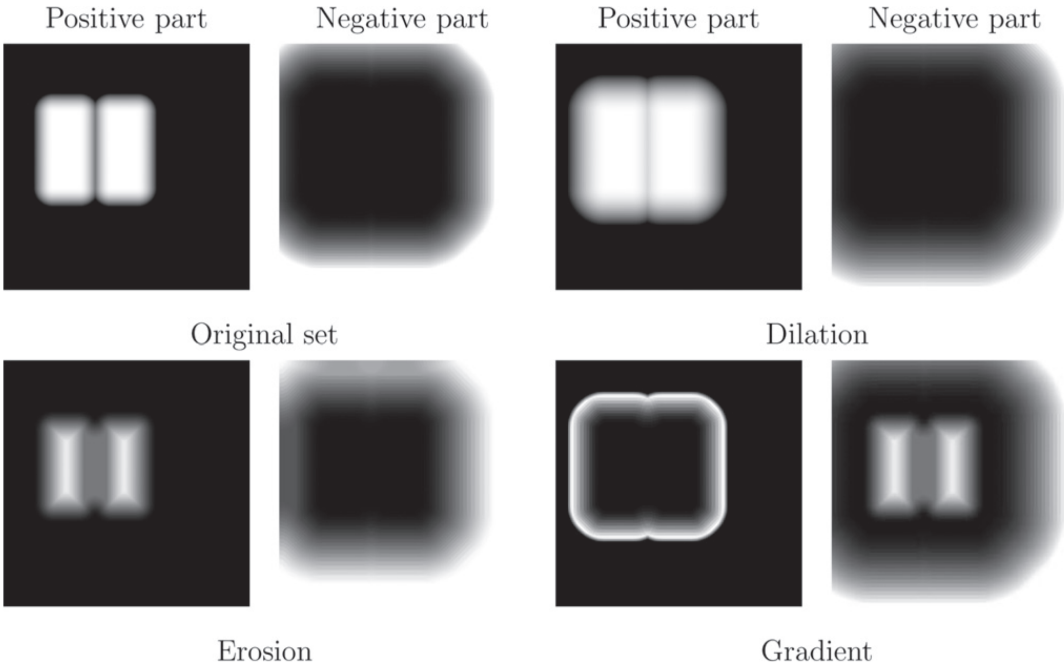

Fig. 6. Bipolar morphological gradient using operators on the lattice $(\mathcal{B}, \preceq_p)$ and t-representable conjunction and implication derived from min and max. The structuring element is as in Fig. 5.

The choice of structuring element has also an influence. In the crisp continuous case, it can be shown that the difference between dilation and erosion tends towards the modulus of the gradient if the size of the structuring element tends towards 0 [91]. In the discrete case, it is then appropriate to use an elementary structuring element (according to the discrete connectivity defined on $\mathcal{S}$), i.e., the central point and its direct neighbors. In the fuzzy and bipolar cases, the structuring element can be somewhat more extended, in order to represent the local spatial imprecision and indetermination. This will lead to a larger gradient.

A direct application of Definition 10 is the computation of the perimeter of a bipolar fuzzy set, defined as a bipolar fuzzy number[4] $|\nabla(\mu, \nu)|$ where the cardinality $|.|$ is defined as proposed in [18][5]:

Definition 11. Let $(\mu, \nu) \in \mathcal{B}$. Its cardinality is defined as: $\forall n, |(\mu, \nu)|(n) = (|\mu|(n), 1 - |1 - \nu|(n))$.

Proposition 21. *The cardinality introduced in Definition 11 is a bipolar fuzzy number on* $\mathbb{N}$.

In the spatial domain, the cardinality can be interpreted as the surface (in 2D) or the volume (in 3D) of the considered bipolar fuzzy set.

Definition 12. Let (μ, ν) be a bipolar fuzzy set. Its perimeter (or surface) is defined as the bipolar fuzzy number $|\nabla(\mu, \nu)|$, where the gradient $\nabla(\mu, \nu)$ is given in Definition 10 and the cardinality $|.|$ in Definition 11.

An example is shown in Fig. 7.

Other geometrical measures have been extended to the bipolar case in [18].

6.2. Conditional operations and reconstruction

Another direct application of the basic operators concerns the notion of conditional dilation (respectively conditional erosion) [91]. These operations are very useful in mathematical morphology in order to constrain an operation to provide a result restricted to some region of space. In the digital case, a conditional dilation can be expressed using the intersection of the usual dilation with an elementary structuring element and the conditioning set. This operation is iterated in order to provide the conditional dilation with a larger structuring element. Iterating this operation until convergence leads to the notion of reconstruction. This operation is typically used in order to gain in robustness in the cases where we have a marker of some objects, and we want to recover the whole objects marked by this marker, and only these objects.

[4] A bipolar fuzzy number is a pair of fuzzy sets μ and ν such that μ and $1 - \nu$ are fuzzy numbers and $\forall a \in \mathbb{R}$ (*or* $\mathbb{N}$), $\mu(\alpha) + \nu(\alpha) \leq 1$. This definition can be relaxed by allowing $1 - \nu$ be a fuzzy interval (i.e., its core is an interval). If both μ and $1 - \nu$ are fuzzy intervals, then (μ, ν) will be called bipolar fuzzy interval.

[5] As advocated in [48,83,84], the cardinality of a fuzzy set can be adequately defined as a fuzzy set on $\mathbb{N}$, and this extends to type-II fuzzy sets [64].

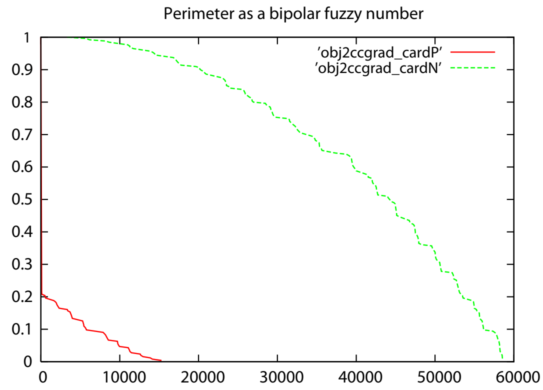

Fig. 7. Perimeter of the bipolar fuzzy set shown in Fig. 6 represented as a bipolar fuzzy number (the negative part is inverted), and computed as the cardinality of the gradient.

Fig. 8. Conditioning set, marker and conditional dilation (only the positive part is shown), on the lattice $(\mathcal{B}, \preceq_p)$, using the Pareto minimum.

The extension of these types of operations to the bipolar fuzzy case is straightforward: given a bipolar fuzzy marker (μ_M, μ_N), the dilation of (μ_M, μ_N), conditionally to a bipolar fuzzy set (μ, ν) is simply defined as the conjunction of the dilation of (μ_M, μ_N) and (μ, ν).

Definition 13. Let (μ, ν) a bipolar fuzzy set and (μ_M, ν_M) a bipolar fuzzy set considered as a marker. The conditional dilation is defined as:

$$\delta((\mu_M, \nu_M)|(\mu, \nu)) = \bigwedge(\delta(\mu_M, \nu_M), (\mu, \nu)). \tag{53}$$

It is easy to show that this defines a bipolar fuzzy set.

In the case of Pareto ordering, this is expressed as: $(\min(\delta^+(\mu_M, \mu_M), \mu), \max(\delta^-(\mu_M, \nu_M), \nu))$, where δ^+ denotes the positive part of the dilation and δ^- its negative part. Since δ^+ can be interpreted as a fuzzy dilation and δ^- as a fuzzy erosion (see Section 4.3), the positive part of the conditional dilation corresponds to a fuzzy conditional dilation of μ (positive part of the initial bipolar fuzzy set), and its negative part corresponds to a fuzzy conditional erosion of ν.

Definition 14. The reconstruction of a bipolar fuzzy set (μ, ν) according to the marker (μ_M, ν_M) is obtained from the iteration of conditional dilations until convergence:

$$R((\mu, \nu), (\mu_B, \nu_B)) = [\delta((\mu_M, \nu_M)|(\mu, \nu))]^\infty. \tag{54}$$

This directly extends the corresponding classical notions in mathematical morphology [91].

An example is shown in Fig. 8, showing that the conditional dilation of the marker is restricted to only one component (the one including the marker) of the original object (only the positive parts are shown). Iterating further this dilation would provide the whole marked component.

Similar definitions can be given for conditional erosion (disjunction with the original bipolar fuzzy set) and reconstruction by erosion.

Note that, to be consistent with the geodesic framework, where the conditional dilation can be expressed according to the geodesic distance in the conditioning set, in the digital case, dilations have to be performed with an elementary structuring element [91]. Here, a crisp non bipolar elementary element can be used as well, but it can be interesting to consider also the smallest bipolar fuzzy structuring element representing local imprecision and bipolarity. This can be further investigated for

Fig. 9. Bipolar fuzzy closing. The fuzzy bipolar structuring element (μ_B, ν_B) of Fig. 5 was used here.

each specific application. By denoting (μ_B, ν_B) this elementary structuring element, the reconstruction is then computed according to the following sequence:

$$\delta^0 = \bigwedge((\mu_M, \nu_M), (\mu, \nu)); \ \delta^1 = \bigwedge(\delta_{(\mu_B,\nu_B)}(\delta^0), (\mu, \nu)) \ \ldots \ \delta^k = \bigwedge(\delta_{(\mu_B,\nu_B)}(\delta^{k-1}), (\mu, \nu)) \ \ldots$$

and the convergence is achieved for n such that $\delta^{n+1} = \delta^n$ (this occurs in a finite number of steps in a discrete bounded (finite) space).

6.3. Opening, closing, and derived operators

In a general algebraic setting, a filter on a complete lattice is defined as an idempotent and increasing operator. An opening γ is an anti-extensive filter and a closing φ is an extensive filter [88].

General properties of γ and φ hold in the lattice $(\mathcal{B}, \preceq)$, as in any complete lattice, whatever the choice of $\preceq$, thanks to the strong algebraic framework and the results of Section 3.1. In particular we have:

- typical examples of opening and closing are $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ where (ε, δ) is an adjunction;
- if (γ_i) is a family of openings, then $\gamma = \vee_i\gamma_i$ is an opening, and if (φ_i) is a family of closings, then $\varphi = \wedge_i\varphi_i$ is a closing;
- by denoting $Inv(\gamma)$ the invariant elements by γ (i.e., bipolar fuzzy sets (μ, ν) such that $\gamma((\mu, \nu)) = (\mu, \nu)$), an opening can be expressed as $\gamma((\mu, \nu)) = \vee\{(\mu', \nu') \in Inv(\gamma) \mid (\mu', \nu') \preceq (\mu, \nu)\}$ [59]. A similar expression holds for φ.

In practice, the morphological forms of erosions and dilations are often used to derive opening and closing. In $(\mathcal{B}, \preceq)$, we have the following monotony properties, for any dilation δ and erosion ε by the same bipolar fuzzy structuring element (omitted in the notations).

Proposition 22. *For any family of bipolar fuzzy sets (μ_i, ν_i), the following inequalities hold:*

$$\vee_i\delta\varepsilon(\mu_i, \nu_i) \preceq \delta\varepsilon(\vee_i(\mu_i, \nu_i)) \quad (55)$$

$$\vee_i\varepsilon\delta(\mu_i, \nu_i) \preceq \varepsilon\delta(\vee_i(\mu_i, \nu_i)) \quad (56)$$

$$\delta\varepsilon(\wedge_i(\mu_i, \nu_i)) \preceq \wedge_i\delta\varepsilon(\mu_i, \nu_i) \quad (57)$$

$$\varepsilon\delta(\wedge_i(\mu_i, \nu_i)) \preceq \wedge_i\varepsilon\delta(\mu_i, \nu_i) \quad (58)$$

These results directly follow from the fact that δ commutes with $\vee$, ε commutes with $\wedge$, and from Proposition 9 (i.e., they hold in general, hence in the particular case of the bipolar setting).

As an example, we consider the lattice $(\mathcal{B}, \preceq_p)$. The closing (obtained using Lukasiewicz operators C_W and D_W) of the bipolar fuzzy object shown in Fig. 6 is displayed in Fig. 9. The small region between the two components in the positive part has been included in this positive part (to some degree) by the closing, which is the expected result.

Another example is shown in Fig. 10, where some small components have been introduced in the bipolar fuzzy set (indicated by circles in the figure, with a high positive information value for one of these additional components, and a low negative information value for the other one). The opening successfully removes these small parts (i.e., small regions with high μ values and small regions with low ν values are removed from the positive part and the negative part, respectively). A typical use of this operation is for situations where the initial bipolar fuzzy set represents possible/forbidden regions for an object. If we have some additional information on the size of the object, so that it is sure that it cannot fit into small parts, then opening can be used to remove possible small places, and to add to the negative part such small regions. This intuitive result is indeed observed in the opening.

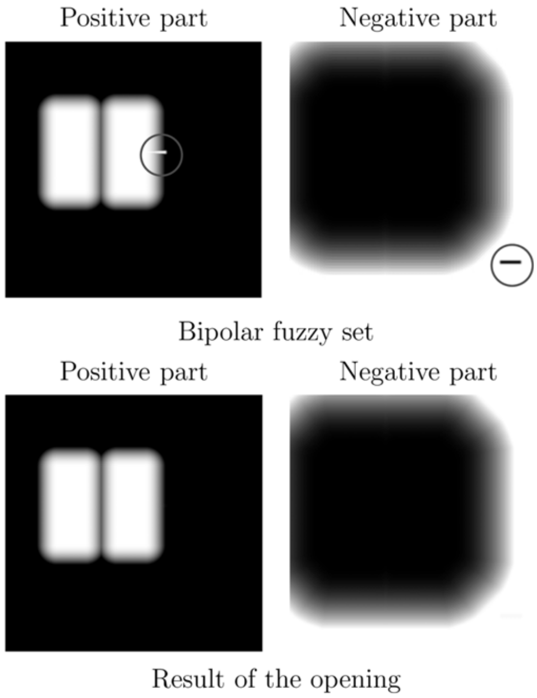

Fig. 10. Bipolar fuzzy opening. A small region with high values of positive information and another one with low values of negative information have been added (indicated by circles). These regions have been removed by the opening (see text). The bipolar fuzzy structuring element (μ_B, ν_B) of Fig. 5 was used in this example.

These operations have simpler expressions if the structuring element is not bipolar (i.e., $\nu_B = 1 - \mu_B$, or $\nu_B = 0$). The positive part of the opening is then the fuzzy opening, using usual Lukasiewicz operators, of μ by μ_B and its negative part is the fuzzy closing of ν by μ_B. Similar equivalences hold for closing.

From these new operators, other ones can be derived, extending the classical ones to the bipolar case. For instance, several filters can be deduced from opening and closing, such as alternate sequential filters [91], by applying alternatively opening and closing, with structuring elements of increasing size. Another example is the top-hat transform [91], which allows extracting bright structures having a given approximative shape, using the difference between the original image and the result of an opening using this shape as a structuring element. Such operators can be directly extended to the bipolar case using the proposed framework.

6.4. Distance from a point to a bipolar fuzzy set

While there is a large amount of work on distances and similarity between interval-valued fuzzy sets or between intuitionistic fuzzy sets (see e.g., [98,101]), none of the existing definitions addresses the question of the distance from a point to a bipolar fuzzy set, nor includes the spatial distance in the proposed definitions, although this is very useful for handling spatial information and for spatial reasoning. As in the fuzzy case [14], we propose to define the distance from a point to a bipolar fuzzy set using a morphological approach. In the crisp case, the distance from a point x to a set X is equal to n iff x belongs to the dilation of size n of X (the dilation of size 0 being the identity), but not to dilations of smaller size (it is sufficient to test this condition for $n-1$ in the discrete case). The transposition of this property to the bipolar fuzzy case leads to the following novel definition, using bipolar fuzzy dilations.

Definition 15. The distance from a point x of $\mathcal{S}$ to a bipolar fuzzy set (μ, ν) $(\in \mathcal{B})$ is defined as:

$$d(x, (\mu, \nu))(0) = (\mu(x), \nu(x)), \tag{59}$$

$$\forall n \in \mathbb{N}^*, d(x, (\mu, \nu))(n) = \delta^n_{(\mu_B, \nu_B)}(\mu, \nu)(x) \wedge N(\delta^{n-1}_{(\mu_B, \nu_B)}(\mu, \nu)(x)), \tag{60}$$

where N is a complementation (typically the standard negation $N(a, b) = (b, a)$ when Pareto ordering is used, or $n_{\preceq_L}$ for lexicographic ordering) and $\delta^n_{(\mu_B, \nu_B)}$ denotes n iterations of the dilation, using the bipolar fuzzy set (μ_B, ν_B) with $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$ as structuring element (to guarantee extensive dilations, see Proposition 14).

In order to clarify the meaning of this definition, let us consider operations on the lattice $(\mathcal{B}, \preceq_p)$ and the case where the structuring element is not bipolar. Then the dilation can be written as: $\delta_{(\mu_B, 1-\mu_B)}(\mu, \nu) = (\delta_{\mu_B}(\mu), \varepsilon_{\mu_B}(\nu))$, where $\delta_{\mu_B}(\mu)$ is the fuzzy dilation of μ by μ_B and $\varepsilon_{\mu_B}(\nu)$ is the fuzzy erosion of ν by μ_B The bipolar degree to which the distance

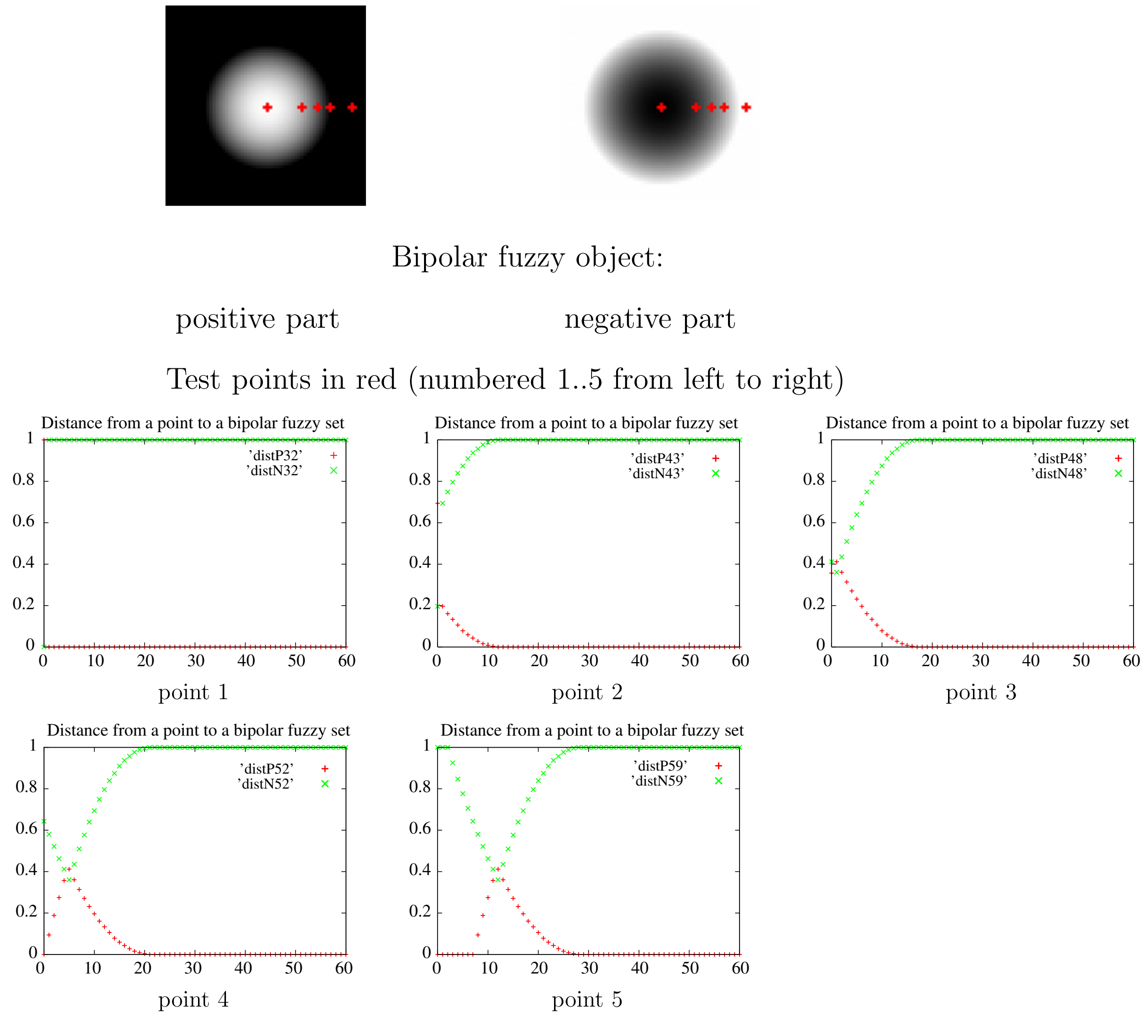

Fig. 11. A bipolar fuzzy set and the distances from five different points to it, represented as bipolar fuzzy numbers (the positive part is shown in red and the negative part in green). (For interpretation of the references to colours in this figure legend, the reader is referred to the web version of this paper.)

from x to (μ, ν) is equal to n is then: $d(x, (\mu, \nu))(n) = (\delta^n_{\mu_B}(\mu) \wedge \varepsilon^{n-1}_{\mu_B}(\nu), \varepsilon^n_{\mu_B}(\nu) \vee \delta^{n-1}_{\mu_B}(\mu))$, i.e., the positive part is the conjunction of the positive part of the dilation of size n (i.e., a dilation of the positive part of the bipolar fuzzy object) and the negative part of the dilation of size $n-1$ (i.e., an erosion of the negative part of the bipolar fuzzy object), and the negative part is the disjunction of the negative part of the dilation of size n (erosion of ν) and the positive part of the dilation of size $n-1$ (dilation of μ).

Proposition 23. *The distance introduced in Definition 15 has the following properties:*

- *it is a bipolar fuzzy set on* $\mathbb{N}$;
- *it reduces to the distance from a point to a fuzzy set, if* (μ, ν) *and* (μ_B, ν_B) *are not bipolar (hence the consistency with the classical definition of the distance from a point to a set is achieved as well);*
- *the distance is strictly equal to 0 (i.e.,* $d(x, (\mu, \nu))(0) = 1_{\mathcal{L}}$ *and* $\forall n \neq 0, d(x, (\mu, \nu))(n) = 0_{\mathcal{L}}$*) iff* $(\mu, \nu)(x) = 1_{\mathcal{L}}$*, i.e., x completely belongs to the bipolar fuzzy set.*

An example is shown in Fig. 11. The results are in agreement with what would be intuitively expected. The positive part of the bipolar fuzzy number is put towards higher values of distances when the point is moved to the right of the object. After a number n of dilations, the point completely belongs to the dilated object, and the value to which the distance is equal to n', with $n' > n$, becomes $0_{\mathcal{L}} = (0, 1)$. Note that the indetermination in the membership or non-membership to the object (which is truly bipolar in this example) is also reflected in the distances.

These distances can be easily compared using the extension principle,[6] providing a bipolar degree $d_{\leq}$ to which a distance is less than another one. For the examples in Fig. 11, we obtain for instance : $d_{\leq}[d(x_1, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.69, 0.20]$

[6] An equivalent of the extension principle is [61,102] $((\mu_1, \nu_1) \circledast (\mu_2, \nu_2))(\gamma) = \vee_{\gamma=\alpha\circledast\beta}(\mu_1, \nu_1)(\alpha) \wedge (\mu_2, \nu_2)(\beta)$, where $\circledast$ denotes any operation. This principle can in particular be applied to define operations on bipolar fuzzy numbers or intervals.

where x_i denotes the ith point from left to right in the figure. In this case, since x_1 completely belongs to (μ, ν), the degree to which its distance is less than the distance from x_2 to (μ, ν) is equal to $[\sup_a d^+(a), \inf_a d^-(a)]$, where d^+ and d^- denote the positive and negative parts of $d(x_2, (\mu, \nu))$. As another example, we have $d_{\leq}[d(x_5, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.03, 0.85]$, reflecting that x_5 is clearly not closer to the bipolar fuzzy set (μ, ν) than x_2.

7. Conclusion

In this paper, we introduced a general algebraic framework for handling bipolar information using mathematical morphology operators. The case of bipolar fuzzy sets has been detailed, since it is general enough to cover several other settings, such as the ones mentioned in Section 2.1 (logical, etc.). Semantics attached to the type of bipolarity have been discussed. This setting, using any partial ordering defining a complete lattice, directly benefits from general properties of mathematical morphology on complete lattices. Two particular orderings have been detailed: Pareto ordering and lexicographic ordering. Their advantages and drawbacks have been discussed, in particular for applications in spatial information processing. Derived operators have been suggested too, thus enlarging the possibilities of handling bipolar spatial information.

Although some preliminary discussion about the type of bipolarity, semantics, orderings, spatial information examples has been suggested, there are still some unanswered questions that deserve further work. The kind of bipolarity depends on the type of information to be handled, and maybe also, to a less extent, on the operations to do so. It would be interesting to elaborate more on the models which are actually relevant for representing and manipulating spatial information, in particular using mathematical morphology tools. Also identifying which ordering is best for what is still an open question (and it depends both on the information, on the way we want to process it, on the objective of this processing, on the expected properties, and on the tools and operations to achieve it). This probably requires to first investigate various applicative problems. One possible direction could be to define criteria according to the problem at hand, and derive a partial ordering accordingly. This has been recently suggested for partition orderings in [87,93], taking into account criteria that are relevant for segmentation purposes.

Examples on the potential use of the new reasoning tools provided by morphological operations have been sketched for both spatial reasoning and preferences modeling. Developing further these applications, along with a deeper investigation of derived operators, with appropriate choices of partial ordering, is the aim of our future work. For instance, when several pieces of information are available, such as information on location, spatial relations, image intensity, shape, they can be combined using fusion tools, in order to get a spatial region accounting for all available information. This type of approach has been used to guide the recognition of anatomical structures in images, based on medical knowledge expressed as a set of spatial relations between pairs or triplets of structures (e.g. in an ontology), in the fuzzy case [20,33,62]. This idea can be extended to the bipolar case. The positive parts can be combined in a conjunctive way and the negative parts in a disjunctive way, according to the semantics of the fusion of bipolar information [47]. This allows reducing the search space for an object by combining spatial relations to reference objects, expressed as bipolar fuzzy sets. This can be considered as an extension to the bipolar case of attention focusing approaches. Illustrations of this idea on the problem of recognition of brain structures from magnetic resonance imaging are presented in [16,18], and the integration of bipolar fuzzy mathematical morphology into descriptions logics for spatial reasoning has been proposed in [63].

Extensions to semi-lattices or general posets could be interestingly considered as well. Concerning the modeling of bipolarity, two ways could be followed: one would be to relax the consistency constraint, so as to directly model potential contradiction or conflict between the two types of information. This would be useful in a number of applications where such conflictual situations may actually occur. Another one could be to model the two types of information using two different orderings (instead of handling them, even potentially asymmetrically, using only one ordering), for instance in the framework of bilattices [54]. This structure was for instance considered in [67] for bipolarity in a logical framework.

Appendix: Proofs of some results

Proof of Proposition 1. It is straightforward to show that the connectives defined by Eqs. (3)–(7) satisfy all required properties, according to Definition 2.

Let us just detail the last two properties, involving the adjunction concept. Let C be a conjunction and I_R defined according to Eq. (8). Then:

$$I_R(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \bigvee\{(a_3, b_3) \in \mathcal{L} \mid C(0_{\mathcal{L}}, (a_3, b_3)) \preceq 0_{\mathcal{L}}\}$$

and since $C(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 0_{\mathcal{L}}$, the supremum is equal to $1_{\mathcal{L}}$.

Since $\forall((a, b), (a_3, b_3)) \in \mathcal{L}^2$, $C((a, b), (a_3, b_3)) \preceq 1_{\mathcal{L}}$, we have:

$$\forall (a, b) \in \mathcal{L}, I_R((a, b), 1_{\mathcal{L}}) = \bigvee\{(a_3, b_3) \in \mathcal{L} \mid C((a, b), (a_3, b_3)) \preceq 1_{\mathcal{L}}\} = 1_{\mathcal{L}}.$$

For the last boundary condition, we have:

$$I_R(1_{\mathcal{L}}, 0_{\mathcal{L}}) = \bigvee\{(a_3, b_3) \in \mathcal{L} \mid C(1_{\mathcal{L}}, (a_3, b_3)) \preceq 0_{\mathcal{L}}\}$$

and since $C(1_{\mathcal{L}}, (a_3, b_3))$ cannot be equal to $0_{\mathcal{L}}$ except for $(a_3, b_3) = 0_{\mathcal{L}}$ by hypothesis, the supremum is equal to $0_{\mathcal{L}}$.

Finally the monotony properties directly result from the ones of C, and hence I_R is an implication.

Let us now show that the adjoint of C is actually I_R, as expressed in Eq. (8). Let (I_R, C) be an adjoint pair. Then, $\forall (a_i, b_i) \in \mathcal{L}, i = 1, \ldots, 3$,

$$C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2) \Leftrightarrow (a_3, b_3) \preceq I((a_1, b_1), (a_2, b_2)).$$

Hence

$$\bigvee\{(a_3, b_3) \in \mathcal{L} \mid C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2)\} \preceq I_R((a_1, b_1), (a_2, b_2)).$$

Conversely, from the tautology $I_R((a_1, b_1), (a_2, b_2)) \preceq I_R((a_1, b_1), (a_2, b_2))$ we derive, by applying the adjunction property and setting $I_R((a_1, b_1), (a_2, b_2)) = (a_3, b_3)$:

$$C((a_1, b_1), I_R((a_1, b_1), (a_2, b_2))) \preceq (a_2, b_2) \Rightarrow$$

$$I_R((a_1, b_1), (a_2, b_2)) \preceq \bigvee\{(a_3, b_3) \mid C((a_1, b_1), (a_3, b_3))) \preceq (a_2, b_2)\}.$$

Hence $I_R = \bigvee\{(a_3, b_3) \in \mathcal{L} \mid C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2)\}$.

The proof for the last property is similar. Note that the condition on I_R makes $C(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$ hold.

Proof of Proposition 5. Let C be a bipolar conjunction that admits $1_{\mathcal{L}}$ as unit element. Since C is increasing, we have:

$$\forall ((a, b), (a', b')) \in \mathcal{L}^2, \;\; C((a, b), (a', b')) \preceq C((a, b), 1_{\mathcal{L}})$$

and $C((a, b), 1_{\mathcal{L}}) = (a, b)$ under the hypothesis. Similarly, $C((a, b), (a', b')) \preceq (a', b')$ and the first result follows.

The two results on I are derived in a similar way, by using the decreasingness of I with respect to the first argument and its increasingness with respect to the second one.

Proof of Proposition 6. Let us first assume that (I, C) is an adjunction. Expressing Eq. (2) for $(a_1, b_1) = (\mu_B, \nu_B)(x - y)$, $(a_3, b_3) = (\mu, \nu)(y)$, $(a_2, b_2)=(\mu', \nu')(x)$ we obtain, for all (μ, ν), (μ', ν'), (μ_B, ν_B) in $\mathcal{B}$, and all x, y in $\mathcal{S}$:

$$C((\mu_B, \nu_B)(x - y), (\mu, \nu)(y)) \preceq (\mu', \nu')(x) \Leftrightarrow (\mu, \nu)(y) \preceq I((\mu_B, \nu_B)(x - y), (\mu', \nu')(x))$$

and, by taking the supremum over y on the left hand side of the equivalence and the infimum over x on the right hand side, we get:

$$\delta_{(\mu_B, \nu_B)}((\mu, \nu)) \preceq (\mu', \nu') \Leftrightarrow (\mu, \nu) \preceq \varepsilon_{(\mu_B, \nu_B)}((\mu', \nu'))$$

hence (ε, δ) is an adjunction.

Conversely, let us assume that (ε, δ) is an adjunction. Expressing the adjunction equivalence for bipolar fuzzy sets taking constant values ($\forall x \in \mathcal{S}$, $(\mu_B, \nu_B)(x) = (a_1, b_1)$, $(\mu, \nu)(x) = (a_2, b_2)$, $(\mu', \nu')(x) = (a_3, b_3)$), we derive immediately the adjunction equivalence for I and C.

Proof of Proposition 8. Let us assume that C and I distribute on the right over the supremum and the infimum respectively. Then for all (μ_B, ν_B) in $\mathcal{B}$ and for any family (μ_i, ν_i) in $\mathcal{B}$, the following equalities hold, $\forall x \in \mathcal{S}$:

$$\begin{aligned}
\delta_{(\mu_B, \nu_B)}(\vee_i(\mu_i, \nu_i))(x) &= \vee_{y \in \mathcal{S}} C((\mu_B, \nu_B)(x - y), \vee_i(\mu_i, \nu_i)(y)) \\
&= \vee_{y \in \mathcal{S}} \vee_i C((\mu_B, \nu_B)(x - y), (\mu_i, \nu_i)(y)) \\
&= \vee_i \delta_{(\mu_B, \nu_B)}((\mu_i, \nu_i))(x) \\
\varepsilon_{(\mu_B, \nu_B)}(\wedge_i(\mu_i, \nu_i))(x) &= \wedge_{y \in \mathcal{S}} I((\mu_B, \nu_B)(y - x), \wedge_i(\mu_i, \nu_i)(y)) \\
&= \wedge_{y \in \mathcal{S}} \wedge_i I((\mu_B, \nu_B)(y - x), (\mu_i, \nu_i)(y)) \\
&= \wedge_i \varepsilon_{(\mu_B, \nu_B)}((\mu_i, \nu_i))(x)
\end{aligned}$$

Hence the distributivity of C and I entails the commutativity of δ with the supremum and the one of ε with the infimum, i.e., δ is a dilation and ε is an erosion.

Conversely, if δ is an algebraic dilation and ε is an algebraic erosion (i.e., they commute with the supremum and the infimum, respectively), then by applying this property to bipolar fuzzy sets taking constant values, the distributivity of C over the supremum on the right and the distributivity of I over the infimum on the right directly follow.

Proof of Proposition 12. The implication directly results from the commutativity of C. The converse implication is obtained by considering constant membership and non-membership functions for the bipolar fuzzy sets.

The proof of Proposition 13 is similar.

Proof of Proposition 14. Let C be a conjunction satisfying the conditions, and assume that $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$. Then, $\forall(\mu_B, \nu_B) \in \mathcal{B}$, $(\mu, \nu) \in \mathcal{B}$, $\forall x \in \mathcal{S}$, since $1_{\mathcal{L}}$ is unit element on the left we have:

$$\begin{aligned}\delta_{(\mu_B,\nu_B)}((\mu, \nu))(x) &\succeq C((\mu_B, \nu_B)(0), (\mu, \nu)(x))\\ &\succeq C(1_{\mathcal{L}}, (\mu, \nu)(x))\\ &\succeq (\mu, \nu)(x)\end{aligned}$$

i.e., δ is extensive.

Conversely, if δ is extensive, let us write the extensivity inequality for the bipolar fuzzy set (μ, ν) defined by $(\mu, \nu)(y) = 0_{\mathcal{L}}$ for $y \neq 0$ and $(\mu, \nu)(0) = 1_{\mathcal{L}}$ and for $x = 0_{\mathcal{L}}$:

$$\begin{aligned}&\vee_y C((\mu_B, \nu_B)(-y), (\mu, \nu)(y)) \succeq (\mu, \nu)(0)\\ &\quad\Rightarrow \vee_{y\neq 0} C((\mu_B, \nu_B)(-y), 0_{\mathcal{L}}) \vee C((\mu_B, \nu_B)(0), 1_{\mathcal{L}}) = 1_{\mathcal{L}}\\ &\quad\Rightarrow 0_{\mathcal{L}} \vee C((\mu_B, \nu_B)(0), 1_{\mathcal{L}}) = C((\mu_B, \nu_B)(0), 1_{\mathcal{L}}) = 1_{\mathcal{L}}\end{aligned}$$

(from the property on the null element). Since under the hypothesis the only value of (a, b) for which $C((a, b), 1_{\mathcal{L}})$ can be equal to $1_{\mathcal{L}}$ is $1_{\mathcal{L}}$, it follows that $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$.

The proof for I and ε is similar.

Proof of Proposition 17. In the fuzzy (non-bipolar case), the Lukasiewicz t-norm is smaller than the product which is smaller than the minimum (largest t-norm). Therefore the inequalities between t-representable bipolar t-norms are straightforward, as well as the one with C_W for its positive part. It is then enough to show the inequality for the negative part of C_W. We have $1 - a_1 \geq b_1$ and $1 - a_2 \geq b_2$ hence $b_2 + 1 - a_1 \geq b_2 + b_1$ and $b_1 + 1 - a_2 \geq b_1 + b_2$. Therefore the negative part of C_W is larger than $\min(1, b_1 + b_2)$, which completes the proof for bipolar t-norms. The reasoning for bipolar t-conorms follows the same line.

Proof of Proposition 19. This proposition is to a large part a particular case of Proposition 5.

In order to show the distributivity property, let us consider any (a, b), (a', b'), (a'', b'') in $\mathcal{L}$, with $(a', b') \preceq_L (a'', b'')$ (the case where the reverse inequality holds is similar). Then:

$$\max_{\preceq_L}((a, b), \min_{\preceq_L}((a', b'), (a'', b''))) = \max_{\preceq_L}((a, b), (a', b'))$$

and, from the increasingness of $\max_{\preceq_L}$:

$$\min_{\preceq_L}(\max_{\preceq_L}((a, b), (a', b')), \max_{\preceq_L}((a, b), (a'', b''))) = \max_{\preceq_L}((a, b), (a', b')).$$

The duality of $\min_{\preceq_L}$ and $\max_{\preceq_L}$ with respect to $n_{\preceq_L}$ is straightforward and directly follows from the fact that $n_{\preceq_L}$ reverses the order.

Proof of Proposition 23. The value at each n is a combination of elements of $\mathcal{L}$, resulting in an element in $\mathcal{L}$. Hence $d(x, (\mu, \nu))$ is a bipolar fuzzy set on $\mathbb{N}$.

Since dilations, negation and infimum reduce to the same notions in the fuzzy case if the set and the structuring element are not bipolar, so is the result (see [22] for the definition of the distance of a point to a fuzzy set using fuzzy dilations).

If $(\mu, \nu)(x) = 1_{\mathcal{L}}$, since $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$, for $n \neq 0$, we have $d(x, (\mu, \nu))(n) = \delta^n_{(\mu_B,\nu_B)}(x) \wedge N(\delta^{n-1}_{(\mu_B,\nu_B)}(x)) = 1_{\mathcal{L}} \wedge 0_{\mathcal{L}} = 0_{\mathcal{L}}$, and $d(x, (\mu, \nu))(0) = 1_{\mathcal{L}}$. Conversely, if $d(x, (\mu, \nu))(0) = 1_{\mathcal{L}}$, then $(\mu, \nu)(x) = d(x, (\mu, \nu))(0) = 1_{\mathcal{L}}$.

References

[1] S. Abramsky, S. Vickers, Quantales, observational logic and process semantics, Mathematical Structures in Computer Science 3 (02) (1993) 161–227.

[2] L. Amgoud, C. Cayrol, M.C. Lagasquie-Schiez, P. Livet, On bipolarity in argumentation frameworks, International Journal of Intelligent Systems 23 (10) (2008) 1062–1093.

[3] J. Angulo, Morphological Colour Operators in Totally Ordered Lattices based on Distance: Application to Image Filtering, Enhancement and Analysis, Computer Vision and Image Understanding 107 (2007) 56–73.

[4] E. Aptoula, S. Lefèvre, A Comparative Study in Multivariate Mathematical Morphology, Pattern Recognition 40 (2007) 2914–2929.
[5] E. Aptoula, S. Lefèvre, On Lexicographic Ordering in Multivariate Mathematical Morphology, Pattern Recognition Letters 29 (2008) 109–118.
[6] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[7] K.T. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper terminology difficulties in fuzzy set theory – the case of intuitionistic fuzzy sets, Fuzzy Sets and Systems 156 (2005) 496–499.
[8] M. Baczynski, On the distributivity of implication operations over t-representable t-norms generated from strict t-norms in interval-valued fuzzy sets theory, in: Information Processing and Management of Uncertainty in Knowledge-Based Systems IPMU 2010, 2010, pp. 637–646.
[9] S. Benferhat, D. Dubois, S. Kaci, H. Prade, Bipolar possibility theory in preference modeling: representation, fusion and optimal solutions, Information Fusion 7 (2006) 135–150.
[10] S. Benferhat, D. Dubois, H. Prade, Modeling positive and negative information in possibility theory, International Journal of Intelligent Systems 23 (10) (2008) 1094–1118.
[11] S. Beucher, J.M. Blosseville, F. Lenoir, Traffic spatial measurements using video image processing, SPIE Conference on Intelligent Robots and Computer Vision 848 (1987) 648–655.
[12] A. Bigand, O. Colot, Fuzzy filter based on interval-valued fuzzy sets for image filtering, Fuzzy Sets and Systems 161 (2010) 96–117.
[13] G. Birkhoff, Lattice Theory, 3rd ed., American Mathematical Society, 1979.
[14] I. Bloch, Fuzzy spatial relationships for image processing and interpretation: a review, Image and Vision Computing 23 (2) (2005) 89–110.
[15] I. Bloch, Dilation and erosion of spatial bipolar fuzzy sets, in: International Workshop on Fuzzy Logic and Applications WILF 2007, vol. LNAI 4578, Genova, Italy, Jul 2007, pp. 385–393.
[16] I. Bloch, Bipolar fuzzy mathematical morphology for spatial reasoning, in: M.H.F. Wilkinson, J.B.T.M. Roerdink, (Eds.), International Symposium on Mathematical Morphology ISMM'09, vol. LNCS 5720, Groningen, The Netherlands, August 2009, pp. 24–34.
[17] I. Bloch, Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations, Fuzzy Sets and Systems 160 (2009) 1858–1867.
[18] I. Bloch, Bipolar fuzzy spatial information: geometry, morphology, spatial reasoning, in: R. Jeansoulin, O. Papini, H. Prade, S. Schockaert (Eds.), Methods for Handling Imperfect Spatial Information, Springer, 2010, pp. 75–102.
[19] I. Bloch, Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology, Information Sciences 181 (2011) 2002–2015.
[20] I. Bloch, T. Géraud, H. Maître, Representation and fusion of heterogeneous fuzzy information in the 3D space for model-based structural recognition – application to 3D brain imaging, Artificial Intelligence 148 (2003) 141–175.
[21] I. Bloch, H. Heijmans, C. Ronse, Mathematical morphology, in: M. Aiello, I. Pratt-Hartman, J. van Benthem (Eds.), Handbook of Spatial Logics, Springer, 2007, pp. 857–947, (Chapter 13).
[22] I. Bloch, H. Maître, Fuzzy mathematical morphologies: a comparative study, Pattern Recognition 28 (9) (1995) 1341–1387.
[23] J.F. Bonnefon, Two routes for bipolar information processing, and a blind spot in between, International Journal of Intelligent Systems 23 (9) (2008) 923–929.
[24] N. Bouaynaya, M. Charif-Chefchaouni, D. Schonfeld, Theoretical foundations of spatially-variant mathematical morphology, Part I: binary images, IEEE Transactions on Pattern Analysis and Machine Intelligence 30 (5) (2008) 823–836.
[25] D. Bouyssou, D. Dubois, M. Pirlot, H. Prade, Concepts and Methods of Decision-Making, ISTE Wiley, 2009.
[26] H. Bustince, E. Barrenechea, M. Pagola, Image thresholding using restricted equivalence functions and maximizing the measures of similarity, Fuzzy Sets and Systems 158 (5) (2007) 496–516.
[27] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Interval-valued fuzzy sets constructed from matrices: application to edge detection, Fuzzy Sets and Systems 160 (13) (2009) 1819–1840.
[28] H. Bustince, P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996) 403–405.
[29] H. Bustince, M. Pagola, E. Barrenechea, Construction of fuzzy indices from fuzzy DI-subsethood measures: application to the global comparison of images, Information Sciences 177 (3) (2007) 906–929.
[30] R. Caferra, N. Peltier, Accepting/rejecting propositions from accepted/rejected propositions: a unifying overview, International Journal of Intelligent Systems 23 (10) (2008) 999–1020.
[31] T. Chaira, A.K. Ray, A new measure using intuitionistic fuzzy set theory and its application to edge detection, Applied Soft Computing Journal 8 (2) (2008) 919–927.
[32] N. Charlier, G. De Tré, S. Gautama, R. Bellens, A twofold fuzzy region model for imprecise quality control of geographic information, in: ICCSA 2008, vol. LNCS 5072, 2008, pp. 647–662.
[33] O. Colliot, O. Camara, I. Bloch, Integration of fuzzy spatial relations in deformable models – application to brain MRI segmentation, Pattern Recognition 39 (2006) 1401–1414.
[34] C. Cornelis, G. Deschrijver, E. Kerre, Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application, International Journal of Approximate Reasoning 35 (2004) 55–95.
[35] C. Cornelis, E. Kerre, Inclusion measures in intuitionistic fuzzy sets, in: ECSQARU'03, vol. LNAI 2711, 2003, pp. 345–356.
[36] P. Couto, H. Bustince, P. Melo-Pinto, M. Pagola, E. Barrenechea, Image segmentation using A-IFSs, in: IPMU'08, Malaga, Spain, 2008, pp. 1620–1627.
[37] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 1990.
[38] B. De Baets, Generalized idempotence in fuzzy mathematical morphology, in: E. Kerre, M. Nachtegael (Eds.), Fuzzy Techniques in Image Processing, Studies in Fuzziness and Soft Computing, vol. 52, Physica Verlag, Springer, 2000, pp. 58–75.
[39] T.-Q. Deng, H. Heijmans, Grey-scale morphology based on fuzzy logic, Journal of Mathematical Imaging and Vision 16 (2002) 155–171.
[40] G. Deschrijver, Triangular norms which are meet-morphisms in intuitionistic fuzzy set theory, Fuzzy Sets and Systems 18 (2011) 88–101.
[41] G. Deschrijver, C. Cornelis, Representability in interval-valued fuzzy set theory, International Journal of Uncertainty Fuzziness and Knowledge Based Systems 15 (3) (2007) 345.
[42] G. Deschrijver, C. Cornelis, E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, IEEE Transactions on Fuzzy Systems 12 (1) (2004) 45–61.
[43] S. Destercke, D. Dubois, E. Chojnacki, Unifying practical uncertainty representations: II. Clouds, International Journal of Approximate Reasoning 49 (3) (2008) 664–677.
[44] D. Dubois, H. Fargier, J.F. Bonnefon, On the qualitative comparison of decisions having positive and negative features, Journal of Artificial Intelligence Research 32 (1) (2008) 385–417.
[45] D. Dubois, H. Fargier, P. Perny, Qualitative decision theory with preference relations and comparative uncertainty: an axiomatic approach, Artificial Intelligence 148 (1) (2003) 219–260.
[46] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade, Terminology difficulties in fuzzy set theory – the case of "intuitionistic fuzzy sets", Fuzzy Sets and Systems 156 (2005) 485–491.
[47] D. Dubois, S. Kaci, H. Prade, Bipolarity in reasoning and decision, an introduction, in: International Conference on Information Processing and Management of Uncertainty, IPMU'04, Perugia, Italy, 2004, pp. 959–966.
[48] D. Dubois, H. Prade, Twofold fuzzy sets abd rough sets – some issues in knowledge representation, Fuzzy Sets and Systems 23 (1987) 3–18.
[49] D. Dubois, H. Prade, An introduction to bipolar representations of information and preference, International Journal of Intelligent Systems 23 (8) (2008) 866–877.
[50] D. Dubois, H. Prade, An overview of the asymmetric bipolar representation of positive and negative information in possibility theory, Fuzzy Sets and Systems 160 (2009) 1355–1366.
[51] H. Fargier, N. Wilson, Algebraic structures for bipolar constraint-based reasoning, in: ECSQARU, vol. LNCS 4724, 2007, pp. 623–634.

[52] M. Galar, J. Fernandez, G. Beliakov, H. Bustince, Interval-valued fuzzy sets applied to stereo matching of color images, IEEE Transactions on Image Processing 20 (7) (2011) 1949–1961.
[53] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, 1980.
[54] M.L. Ginsberg, Multivalued logics: a uniform approach to reasoning in artificial intelligence, Computational intelligence 4 (3) (1988) 265–316.
[55] J.A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1) (1967) 145–174.
[56] M. Grabisch, S. Greco, M. Pirlot, Bipolar and bivariate models in multicriteria decision analysis: descriptive and constructive approaches, International Journal of Intelligent Systems 23 (9) (2008) 930–969.
[57] J.G. Gutiérrez-García, S.E. Rodabaugh, Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued, Fuzzy Sets and Systems 156 (3) (2005) 445–484.
[58] S. Hazarika, A.G. Cohn, A taxonomy for spatial vagueness: an alternative egg-yolk interpretation, in: Spatial Vagueness, Uncertainty and Granularity Symposium, Ogunquit, Maine, USA, 2001.
[59] H. Heijmans, Morphological Image Operators, Academic Press, Boston, 1994.
[60] H.J.A.M. Heijmans, C. Ronse, The algebraic basis of mathematical morphology – Part I: dilations and erosions, Computer Vision, Graphics and Image Processing 50 (1990) 245–295.
[61] D.H. Hong, S. Lee, Some algebraic properties and a distance measure for interval-valued fuzzy numbers, Information Sciences 148 (1–4) (2002) 1–10.
[62] C. Hudelot, J. Atif, I. Bloch, Fuzzy Spatial Relation Ontology for Image Interpretation, Fuzzy Sets and Systems 159 (2008) 1929–1951.
[63] C. Hudelot, J. Atif, I. Bloch, Integrating bipolar fuzzy mathematical morphology in description logics for spatial reasoning, in: European Conference on Artificial Intelligence ECAI 2010, Lisbon, Portugal, August 2010, pp. 497–502.
[64] L.C. Jang, D. Ralescu, Cardinality concepts for type-two fuzzy sets, Fuzzy Sets and Systems 118 (3) (2001) 479–487.
[65] S. Kaci, Logical formalisms for representing bipolar preferences, International Journal of Intelligent Systems 23 (8) (2008) 985–997.
[66] R. Keshet, Mathematical morphology on complete semilattices and its applications to image processing, Fundamenta Informaticae 41 (2000) 33–56.
[67] S. Konieczny, P. Marquis, P. Besnard, Bipolarity in bilattice logics, International Journal of Intelligent Systems 23 (10) (2008) 1046–1061.
[68] F. Lange, M. Grabisch, New axiomatizations of the Shapley interaction index for bi-capacities, Fuzzy Sets and Systems 176 (2011) 64–75.
[69] M.R. Malek, Spatial object modeling in intuitionistic fuzzy topological spaces, in: Rough Sets And Current Trends In Computing: 4th International Conference, RSCTC 2004, vol. LNAI 3066, Uppsala, Sweden, 2004, pp. 427–434.
[70] M.R. Malek, Intuitionistic fuzzy spatial relationships in mobile GIS environment, in: International Workshop on Fuzzy Logic and Applications WILF 2007, vol. LNAI 4578, 2007, pp. 313–320.
[71] P. Maragos, Lattice image processing: a unification of morphological and fuzzy algebraic systems, Journal of Mathematical Imaging and Vision 22 (2005) 333–353.
[72] J. Medina, Adjoint pairs on interval-valued fuzzy sets, in: Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2010, 2010, pp. 430–439.
[73] T. Mélange, M. Nachtegael, P. Sussner, E. Kerre, Basic properties of the interval-valued fuzzy morphological operators, in: IEEE World Congress on Computational Intelligence WCCI 2010, Barcelona, Spain, 2010, pp. 822–829.
[74] T. Mélange, M. Nachtegael, P. Sussner, E. Kerre, On the decomposition of interval-valued fuzzy morphological operators, Journal of Mathematical Imaging and Vision 36 (2010) 270–290.
[75] T. Mélange, M. Nachtegael, P. Sussner, E. Kerre, On the construction of interval-valued fuzzy morphological operators, Fuzzy Sets and Systems 178 (1) (2011) 84–101.
[76] H. Moulin, Axioms of Cooperative Decision Making, Cambridge University Press, 1991.
[77] M. Nachtegael, E.E. Kerre, Classical and fuzzy approaches towards mathematical morphology, in: E.E. Kerre, M. Nachtegael (Eds.), Fuzzy Techniques in Image Processing, Studies in Fuzziness and Soft Computing, Physica-Verlag, Springer, 2000, pp. 3–57, (Chapter 1).
[78] M. Nachtegael, P. Sussner, T. Mélange, E. Kerre, Some aspects of interval-valued and intuitionistic fuzzy mathematical morphology, in: IPCV 2008, 2008.
[79] M. Nachtegael, P. Sussner, T. Melange, E.E. Kerre, On the role of complete lattices in mathematical morphology: from tool to uncertainty model, Information Sciences 181 (2011) 1971–1988.
[80] A. Neumaier, Clouds, fuzzy sets, and probability intervals, Reliable Computing 10 (4) (2004) 249–272.
[81] R. Da Silva Neves, P. Livet, Bipolarity in human reasoning and affective decision making, International Journal of Intelligent Systems 23 (8) (2008) 898–922.
[82] M. Öztürk, A. Tsoukias, Bipolar preference modeling and aggregation in decision support, International Journal of Intelligent Systems 23 (9) (2008) 970–984.
[83] A.L. Ralescu, A note on rule representation in expert systems, Information Sciences 38 (2) (1986) 193–203.
[84] D. Ralescu, Cardinality, quantifiers, and the aggregation of fuzzy criteria, Fuzzy Sets and Systems 69 (3) (1995) 355–365.
[85] E. Raufaste, S. Vautier, An evolutionist approach to information bipolarity: representations and affects in human cognition, International Journal of Intelligent Systems 23 (8) (2008) 878–897.
[86] C. Ronse, Why mathematical morphology needs complete lattices, Signal Processing 21 (2) (1990) 129–154.
[87] C. Ronse, Orders on partial partitions and maximal partitioning of sets, Mathematical Morphology and Its Applications to Image and Signal Processing, LNCS 6671, 2011, pp. 49–60.
[88] C. Ronse, H.J.A.M. Heijmans, The algebraic basis of mathematical morphology – part II: openings and closings, Computer Vision, Graphics and Image Processing 54 (1991) 74–97.
[89] B. Roy, The outranking approach and the foundations of ELECTRE methods, Theory and Decision 31 (1) (1991) 49–73.
[90] C. Russo, A unified algebraic framework for fuzzy image compression and mathematical morphology, in: International Workshop on Fuzzy Logic and Applications (WILF 2009), vol. LNAI 5571, 2009, pp. 205–212.
[91] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.
[92] J. Serra, Image Analysis and Mathematical Morphology, Part II: Theoretical Advances, Academic Press (J. Serra Ed.), London, 1988.
[93] J. Serra, Grain building ordering, Mathematical Morphology and Its Applications to Image and Signal Processing, LNCS 6671, 2011, pp. 37–48.
[94] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: theory and applications, Fuzzy Sets and Systems 55 (1993) 15–42.
[95] P. Soille, Morphological Image Analysis, Springer-Verlag, Berlin, 1999.
[96] J.G. Stell, Why mathematical morphology needs quantales, in: M. Wilkinson, J. Roerdink (Eds.), Abstract book of the 9th Int. Symp on Math. Morph (ISMM'2009), The Netherlands, 2009, University of Groningen, pp. 13–16.
[97] P. Sussner, M. Nachtegael, T. Mélange, G. Deschrijver, E. Esmi, E. Kerre, Interval-valued and intuitionistic fuzzy mathematical morphologies as special cases of L-fuzzy mathematical morphology, Journal of Mathematical Imaging and Vision 43 (1) (2012) 50–71.
[98] E. Szmidt, J. Kacprzyk, Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (3) (2000) 505–518.
[99] H. Talbot, C. Evans, R. Jones, Complete ordering and multivariate mathematical morphology, in: Proceedings of the fourth international symposium on Mathematical morphology and its applications to image and signal processing, 1998, pp. 27–34.
[100] L. Vincent, Graphs and mathematical morphology, Signal Processing 16 (1989) 365–388.
[101] I.K. Vlachos, G.D. Sergiadis, Intuitionistic fuzzy information – applications to pattern recognition, Pattern Recognition Letters 28 (2) (2007) 197–206.
[102] G. Wang, X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems 98 (3) (1998) 331–335.
[103] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Information Sciences 8 (1975) 199–249.
[104] W.R. Zhang, L. Zhang, YinYang bipolar logic and bipolar fuzzy logic, Information Sciences 165 (3–4) (2004) 265–287.