Bipolar Fuzzy Spatial Information: Geometry, Morphology, Spatial Reasoning

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Abstract. Spatial information may be endowed with a bipolarity component. Typical examples concern possible vs forbidden places for an object in space, or "opposite" spatial relations such as "possibly to the right of an object and certainly not to its left". However, bipolarity has not been much exploited in the spatial domain yet. Moreover, imprecision has often to be taken into account as well, for instance to model vague statements such as "to the right of an object". In this paper we propose to handle both features in the framework of bipolar fuzzy sets. We introduce some geometrical measures and mathematical morphology operations on bipolar fuzzy sets and illustrate their potential for spatial reasoning on a simple scenario in brain imaging.

1 Introduction

In many domains, it is important to be able to deal with bipolar information [34, 36, 37]. Positive information represents what is possible, for instance because it has already been observed or experienced, while negative information represents what is impossible or forbidden, or surely false. The intersection of the positive information and the negative information has to be empty in order to achieve consistency of the representation, and their union does not necessarily cover the whole underlying space, i.e. there is no direct duality between both types of information.

This domain has recently motivated work in several directions, for instance for applications in knowledge representation, preference modeling, argumentation, multi-criteria decision analysis, cooperative games, among others [1, 5, 21, 23, 37, 38, 42, 43, 49, 50, 51]. In particular, fuzzy and possibilistic formalisms for

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bipolar information have been proposed [4, 5, 34, 36]. Interestingly enough, they are formally linked to intuitionistic fuzzy sets [2], interval-valued fuzzy sets [60] and vague sets, as shown by several authors [22, 33]. However, their respective semantics differ.

When dealing with spatial information, in image processing or for spatial reasoning applications, this bipolarity also occurs. For instance, when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider “left” and “right” as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of “opposite” captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indetermination [6]. This corresponds to the idea that the union of positive and negative information does not cover all the space. Similar considerations can be provided for other pairs of “opposite” relations, such as “close to” and “far from” for instance.

An example is illustrated in Figure 1. It shows an object at some position in the space (the rectangle in this figure). Let us assume that some information about the position of another object is provided: it is to the left of the rectangle and not to the right. The region “to the left of the rectangle” is computed using a fuzzy dilation with a directional fuzzy structuring element providing the semantics of “to the left” [6], thus defining the positive information. The region “to the right of the rectangle” defines the negative information and is computed in a similar way. The membership functions μ_L and μ_R represent respectively the positive and negative parts of the bipolar fuzzy set. They are not the complement of each other, and we have: $\forall x, \mu_L(x) + \mu_R(x) \leq 1$.

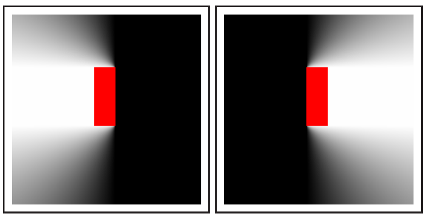

Fig. 1 Region to the left of the rectangle (positive information, μ_L) and region to the right of the rectangle (negative information, μ_R). The membership degrees vary from 0 (black) to 1 (white).

Another example, for the pair of relations close/far, is illustrated in Figure 2. The reference object is the square in the center of the image. The two fuzzy regions are computed using fuzzy dilations, using structuring elements that provide the semantics of “close” and “far” [7]. Again, the two membership functions μ_C and μ_F are

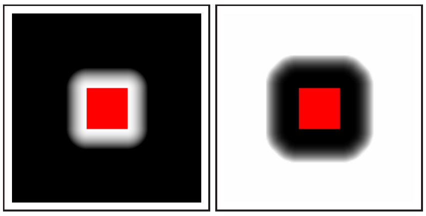

Fig. 2 Region close to the square (μ_C) and region far from the square (μ_F)

not the complement of each other and actually define a bipolar fuzzy set, with its positive and negative parts.

To our knowledge, bipolarity has not been much exploited in the spatial domain. A few works deal with image thresholding or edge detection, based on intuitionistic fuzzy sets derived from image intensity and entropy or divergence criteria [24, 29, 57]. Spatial representations of interval-valued fuzzy sets have also been proposed in [25], as a kind of fuzzy egg-yolk, for evaluating classification errors based on ground-truth, or in [44, 45] with preliminary extensions of RCC to these representations. But there are still very few tools for manipulating spatial information using both its bipolarity (and not simply some kind of imprecision on the membership values) and imprecision components.

The above considerations are the motivation for the present work, which aims at filling this gap by proposing formal models to manage spatial bipolar information. We consider here both objects and spatial relations between objects, as motivated by the previous examples. Additionally, imprecision has to be included, since it is an important feature of spatial information, related either to the objects themselves or to the spatial relations between them. For spatial relations, we consider their spatial representations, as proposed in [8], defining the regions of space where a relation to a reference object is satisfied (to some degree). More specifically, we consider bipolar fuzzy sets in the spatial domain, representing either objects or spatial relations to some reference objects, and propose definitions of some geometrical measures and of mathematical morphology operators (dilation and erosion) on these representations, extending our preliminary work [12, 14]. The choice of mathematical morphology for a first insight into the manipulation of spatial bipolar fuzzy sets is related to its wide use in image and spatial information processing [52, 54], its interest for modeling spatial relations in various formal settings (quantitative, qualitative, or fuzzy) [11], and its strong algebraic basis [39].

In Section 2, we recall some definitions on bipolar fuzzy sets. Then we introduce definitions of some simple geometrical measures on spatial bipolar fuzzy sets, in Section 3. In Section 4, we extend our work on mathematical morphology and detail definitions of erosion and dilation using a bipolar fuzzy structuring element, their properties, and some derived operations. Finally, in Section 5, we suggest some ways to define bipolar fuzzy representations of spatial relations, and we present some examples for spatial reasoning in Section 6.

2 Background

Let $\mathscr{S}$ be the underlying space (the spatial domain for spatial information processing), that is supposed to be bounded and finite here. A bipolar fuzzy set on $\mathscr{S}$ is defined by a pair of functions (μ, ν) such that $\forall x \in \mathscr{S}, \mu(x) + \nu(x) \leq 1$. Note that a bipolar fuzzy set is formally (although not semantically) equivalent to an intuitionistic fuzzy set [2]. It is also equivalent to an interval-valued fuzzy set [60], where the interval at each point x is $[\mu(x), 1 - \nu(x)]$ [33]. Although there has been a lot of discussion about terminology in this domain recently [3, 33], we use the bipolarity terminology in this paper, for its appropriate semantics, as explained in our motivation. For each point x, $\mu(x)$ defines the degree to which x belongs to the bipolar fuzzy set (positive information) and $\nu(x)$ the non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. Concerning semantics, it should be noted that a bipolar fuzzy set does not necessarily represent one physical object or spatial entity, but rather more complex information, potentially issued from different sources.

Let us consider the set $\mathscr{L}$ of pairs of numbers (a,b) in $[0,1]$ such that $a+b \leq 1$. This set is a complete lattice, for the partial order defined as [28]:

$$(a_1, b_1) \preceq (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \geq b_2. \tag{1}$$

The greatest element is $(1,0)$ and the smallest element is $(0,1)$. The supremum and infimum are respectively defined as:

$$(a_1, b_1) \vee (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2)), \tag{2}$$

$$(a_1, b_1) \wedge (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2)). \tag{3}$$

The partial order $\preceq$ induces a partial order on the set of bipolar fuzzy sets:

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \text{ iff } \forall x \in \mathscr{S}, \mu_1(x) \leq \mu_2(x) \text{ and } \nu_1(x) \geq \nu_2(x). \tag{4}$$

Note that this corresponds to the inclusion on intuitionistic fuzzy sets [2]. Similarly the supremum and the infimum are equivalent to the intuitionistic union and intersection.

It follows that, if $\mathscr{B}$ denotes the set of bipolar fuzzy sets on $\mathscr{S}$, $(\mathscr{B}, \preceq)$ is a complete lattice.

3 Some Basic Geometrical Measures

3.1 Cardinality

Let $(\mu, \nu) \in \mathscr{B}$ be a bipolar fuzzy set defined in the spatial domain $\mathscr{S}$. The cardinality of intuitionistic or interval valued fuzzy sets has been introduced e.g. in [56] as an interval: $[\sum_{x \in \mathscr{S}} \mu(x), \sum_{x \in \mathscr{S}} (1 - \nu(x))]$, with the lower bound representing the classical cardinality of the fuzzy set defining the positive part (the least certain cardinality), and the upper bound the cardinality of the complement of the negative part (i.e. the whole not impossible region is considered, leading to the largest possible cardinality). The length of the interval reflects the indetermination encoded by the bipolar representation. Several authors have used a similar approach, based on interval representations of the cardinality.

When dealing with fuzzy sets, it may be more interesting to consider the cardinality as a fuzzy number, instead as a crisp number, for instance using the extension principle [35]: $|\mu|(n) = \sup\{\alpha \in [0,1] \mid |\mu_\alpha| = n\}$, where μ_α denotes α-cuts, defining the degree to which the cardinality of μ is equal to n.

Here we propose a similar approach for defining the cardinality of a bipolar fuzzy set as a bipolar fuzzy number, which contrasts with the previously interval-based approaches.

Definition 1. *Let $(\mu, \nu) \in \mathscr{B}$. Its cardinality is defined as:*

$$\forall n, |(\mu, \nu)|(n) = (|\mu|(n), 1 - |1 - \nu|(n)). \tag{5}$$

Proposition 1. *The cardinality introduced in Definition 1 is a bipolar fuzzy number, i.e. a bipolar fuzzy set defined on $\mathbb{N}$, with $\forall n, |\mu|(n) + (1 - |1 - \nu|(n)) \leq 1$.*

In the spatial domain, the cardinality can be interpreted as the surface (in 2D) or the volume (in 3D) of the considered bipolar fuzzy set. Let us consider the example of possible/forbidden places for an object, represented by (μ, ν). Then the positive part of the cardinality represents how large is the possible set of places, while the negative part is linked to the size of the forbidden regions.

An example is shown in Figure 3. For this example, the cardinality computed as an interval would provide $[11000, 40000]$, which approximately corresponds to the 0.5-level of the bipolar fuzzy number.

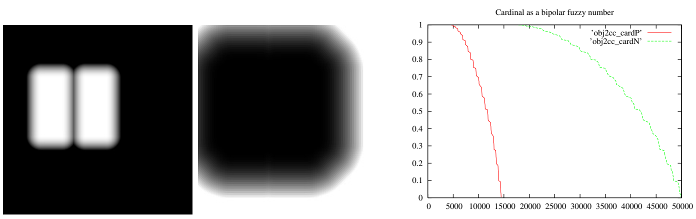

Fig. 3 Bipolar fuzzy set (positive part and negative part) and its cardinality represented as a bipolar fuzzy number (the negative part, in green, is inverted)

3.2 Center of Gravity

Here we propose a simple approach to define the center of gravity of a bipolar fuzzy set, that accounts for the indetermination. The underlying idea is that a point should contribute a lot if it belongs strongly to μ (positive part) and weakly to ν

(negative part). Just translating this idea as a weight for each point $x \in \mathscr{S}$ defined as $\min(\mu(x), 1-\nu(x))$ is not interesting since this always provides $\mu(x)$ and the indetermination is then not taken into account.

Therefore, we define the center of gravity by weighting each point by its membership to the positive part plus a portion of the indetermination.

Definition 2. *The center of gravity of a bipolar fuzzy set* $(\mu,\nu) \in \mathscr{B}$ *is defined as:*

$$CoG(\mu,\nu) = \frac{\sum_{x\in\mathscr{S}} x(\mu(x)+\lambda\pi(x))}{\sum_{x\in\mathscr{S}}(\mu(x)+\lambda\pi(x))} \tag{6}$$

with $\pi(x) = 1-\mu(x)-\nu(x)$ *denotes the indetermination and* λ *is a weighting factor, with* $\lambda \in [0,1]$.

The parameter λ allows tuning the influence of the indetermination. For $\lambda = 0$, Definition 2 leads to the classical center of gravity of a fuzzy set, by considering only the positive part μ. For $\lambda = 1$, it leads to the center of gravity of $1-\nu$ (i.e. everything that is not impossible is included in the computation). Intermediate values of λ realize a gradual compromise between these two extreme solutions. This is illustrated in Figure 4.

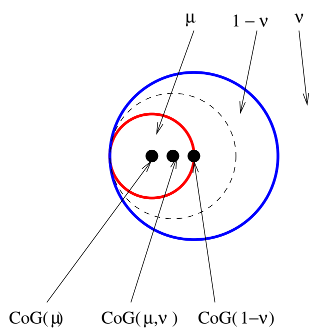

Fig. 4 A spatial bipolar set (crisp in this example) and its center of gravity for $\lambda = 0.5$ (corresponding to the center of the dashed circle in this case)

Other moments could be defined in a similar way.

It should be noted that this approach is mainly relevant if the bipolar fuzzy set is considered as one spatial entity, which is not always the case, as mentioned in Section 2. In cases where the bipolar fuzzy set represents some more complex information, pertaining to a same situation but potentially representing different pieces of information or knowledge coming from different sources, then the meaning itself of a center of gravity has to be reconsidered.

4 Mathematical Morphology

Mathematical morphology on bipolar fuzzy sets has been first introduced in [12]. Once we have a complete lattice, as described in Section 2, it is easy to define algebraic dilations and erosions on this lattice, as operators that commute with the supremum and the infimum, respectively:

$$\delta((\mu,\nu)\vee(\mu',\nu')) = \delta((\mu,\nu))\vee\delta((\mu',\nu')), \tag{7}$$

$$\varepsilon((\mu,\nu)\wedge(\mu',\nu')) = \varepsilon((\mu,\nu))\wedge\varepsilon((\mu',\nu')), \tag{8}$$

and similar expressions for sup and inf taken over any family of bipolar fuzzy sets. Their properties are derived from general properties of lattice operators. If we assume that $\mathscr{S}$ is an affine space (or at least a space on which translations can be defined), it is interesting, for dealing with spatial information, to consider morphological operations based on a structuring element, which are hence invariant under translation. A structuring element is a subset of $\mathscr{S}$ with fixed shape and size, directly influencing the spatial extent of the morphological transformations. It is generally assumed to be compact, so as to guarantee good properties. In the discrete case considered here, we assume that it is connected, in the sense of a discrete connectivity defined on $\mathscr{S}$. The general principle underlying morphological operators consists in translating the structuring element at every position in space and checking if this translated structuring element satisfies some relation with the original set (inclusion for erosion, intersection for dilation) [52]. This principle has also been used in the main extensions of mathematical morphology to fuzzy sets [20, 30, 31, 46, 47, 53]. We detail the construction of such morphological operators, extending our preliminary work [12, 13], along with some derived operators.

4.1 Erosion

As for fuzzy sets [20], defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets [32]. With our notations, a degree of inclusion of a bipolar fuzzy set (μ',ν') in another bipolar fuzzy set (μ,ν) is defined as:

$$\inf_{x\in\mathscr{S}} I((\mu'(x),\nu'(x)),(\mu(x),\nu(x))) \tag{9}$$

where I is an implication operator. Two types of implication are considered [27, 32], among the different classes of implications, one derived from a bipolar t-conorm $\perp$[1]:

[1] A bipolar disjunction is an operator D from $\mathscr{L}\times\mathscr{L}$ into $\mathscr{L}$ such that $D((1,0),(1,0)) = D((0,1),(1,0)) = D((1,0),(0,1)) = (1,0)$, $D((0,1),(0,1)) = (0,1)$ and that is increasing in both arguments. A bipolar t-conorm is a commutative and associative bipolar disjunction such that the smallest element of $\mathscr{L}$ is the unit element.

$$I_N((a_1,b_1),(a_2,b_2)) = \perp((b_1,a_1),(a_2,b_2)), \tag{10}$$

and one derived from a residuation principle from a bipolar t-norm $\top$[2]:

$$I_R((a_1,b_1),(a_2,b_2)) = \sup\{(a_3,b_3) \in \mathscr{L} \mid \top((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2)\} \tag{11}$$

where $(a_i,b_i) \in \mathscr{L}$ and (b_i,a_i) is the standard negation of (a_i,b_i).

Two types of t-norms and t-conorms are considered in [32] and will be considered here as well:

1. operators called t-representable t-norms and t-conorms, which can be expressed using usual t-norms t and t-conorms T from the fuzzy sets theory [35]:

$$\top((a_1,b_1),(a_2,b_2)) = (t(a_1,a_2), T(b_1,b_2)), \tag{12}$$

$$\perp((a_1,b_1),(a_2,b_2)) = (T(a_1,a_2), t(b_1,b_2)). \tag{13}$$

2. Lukasiewicz operators, which are not t-representable:

$$\top_W((a_1,b_1),(a_2,b_2)) = (\max(0,a_1+a_2-1), \min(1,b_1+1-a_2,b_2+1-a_1)), \tag{14}$$

$$\perp_W((a_1,b_1),(a_2,b_2)) = (\min(1,a_1+1-b_2,a_2+1-b_1), \max(0,b_1+b_2-1)). \tag{15}$$

In these equations, the positive part of $\top_W$ is the usual Lukasiewicz t-norm of a_1 and a_2 (i.e. the positive parts of the input bipolar values). The negative part of $\perp_W$ is the usual Lukasiewicz t-norm of the negative parts (b_1 and b_2) of the input values. The two types of implication coincide for the Lukasiewicz operators [28].

Based on these concepts, we can now propose a definition for morphological erosion.

Definition 3. *Let (μ_B,ν_B) be a bipolar fuzzy structuring element (in $\mathscr{B}$). The erosion of any (μ,ν) in $\mathscr{B}$ by (μ_B,ν_B) is defined from an implication I as:*

$$\forall x \in \mathscr{S}, \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \inf_{y\in\mathscr{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y))), \tag{16}$$

where $\mu_B(y-x)$ denotes the value at point y of μ_B translated at x.

A similar approach has been used for intuitionistic fuzzy sets in [48], but with weaker properties (in particular an important property such as the commutativity of erosion with the conjunction may be lost).

[2] A bipolar conjunction is an operator C from $\mathscr{L} \times \mathscr{L}$ into $\mathscr{L}$ such that $C((0,1),(0,1)) = C((0,1),(1,0)) = C((1,0),(0,1)) = (0,1)$, $C((1,0),(1,0)) = (1,0)$ and that is increasing in both arguments. A bipolar t-norm is a commutative and associative bipolar conjunction such that the largest element of $\mathscr{L}$ is the unit element.

4.2 Morphological Dilation of Bipolar Fuzzy Sets

Dilation can be defined based on a duality principle or based on the adjunction property. Both approaches have been developed in the case of fuzzy sets, and the links between them and the conditions for their equivalence have been proved in [9, 16]. Similarly we consider both approaches to define morphological dilation on $\mathscr{B}$.

4.2.1 Dilation by Duality

The duality principle states that the dilation is equal to the complementation of the erosion, by the same structuring element (if it is symmetrical with respect to the origin of $\mathscr{S}$, otherwise its symmetrical is used), applied to the complementation of the original set. Applying this principle to bipolar fuzzy sets using a complementation c (typically the standard negation $c((a,b)) = (b,a)$) leads to the following definition of morphological bipolar dilation.

Definition 4. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element. The dilation of any (μ, ν) in $\mathscr{B}$ by (μ_B, ν_B) is defined from erosion by duality as:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu)) = c[\varepsilon_{(\mu_B,\nu_B)}(c((\mu,\nu)))]. \tag{17}$$

4.2.2 Dilation by Adjunction

Let us now consider the adjunction principle, as in the general algebraic case. An adjunction property can also be expressed between a bipolar t-norm and the corresponding residual implication as follows:

$$\top((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2) \Leftrightarrow (a_3,b_3) \preceq I_R((a_1,b_1),(a_2,b_2)). \tag{18}$$

Definition 5. *Using a residual implication for the erosion for a bipolar t-norm $\top$, the bipolar fuzzy dilation, adjoint of the erosion, is defined as:*

$$\begin{aligned}\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \inf\{(\mu',\nu')(x) \mid (\mu,\nu)(x) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu'))(x)\} \\ &= \sup_{y\in\mathscr{S}} \top((\mu_B(x-y),\nu_B(x-y)),(\mu(y),\nu(y))). \end{aligned} \tag{19}$$

4.2.3 Links between Both Approaches

It is easy to show that the bipolar Lukasiewicz operators are adjoint, according to Equation 18. It has been shown that the adjoint operators are all derived from the Lukasiewicz operators, using a continuous bijective permutation on $[0,1]$ [32]. Hence equivalence between both approaches can be achieved only for this class of operators. This result is similar to the one obtained for fuzzy mathematical morphology [9, 16].

4.3 Properties

Proposition 2. *All definitions are consistent: they actually provide bipolar fuzzy sets of* $\mathscr{B}$.

Proposition 3. *In case the bipolar fuzzy sets are usual fuzzy sets (i.e.* $\nu = 1-\mu$ *and* $\nu_B = 1-\mu_B$*), the definitions lead to the usual definitions of fuzzy dilations and erosions (using classical Lukasiewicz t-norm and t-conorm for the definitions based on the Lukasiewicz operators). Hence they are also compatible with classical morphology in case* μ *and* μ_B *are crisp.*

Proposition 4. *The proposed definitions of bipolar fuzzy dilations and erosions commute respectively with the supremum and the infinum of the lattice* $(\mathscr{B}, \preceq)$.

Proposition 5. *The bipolar fuzzy dilation is extensive (i.e.* $(\mu,\nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu))$*) and the bipolar fuzzy erosion is anti-extensive (i.e.* $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu,\nu)$*) if and only if* $(\mu_B,\nu_B)(0) = (1,0)$*, where* 0 *is the origin of the space* $\mathscr{S}$ *(i.e. the origin completely belongs to the structuring element, without any indetermination).*

Note that this condition is equivalent to the conditions on the structuring element found in classical and fuzzy morphology to have extensive dilations and anti-extensive erosions [20, 52].

Proposition 6. *The dilation satisfies the following iterativity property:*

$$\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}((\mu,\nu))) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu,\nu)). \tag{20}$$

Proposition 7. *Conversely, if we want all classical properties of mathematical morphology to hold true, the bipolar conjunctions and disjunctions used to define intersection and inclusion in* $\mathscr{B}$ *have to be bipolar t-norms and t-conorms. If both duality and adjunction are required, then the only choice is bipolar Lukasiewicz operators (up to a continuous permutation on* $[0,1]$*).*

This result [15] is very important, since it shows that the proposed definitions are the most general ones to have a satisfactory interpretation in terms of mathematical morphology.

4.4 Interpretations

Let us first consider the implication defined from a t-representable bipolar t-conorm. Then the erosion is written as:

$$\begin{aligned}\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \inf_{y\in\mathscr{S}} \perp((\nu_B(y-x),\mu_B(y-x)),(\mu(y),\nu(y))) \\ &= (\inf_{y\in\mathscr{S}} T(\nu_B(y-x),\mu(y)), \sup_{y\in\mathscr{S}} t(\mu_B(y-x),\nu(y))). \end{aligned} \tag{21}$$

This resulting bipolar fuzzy set has a membership function which is exactly the fuzzy erosion of μ by the fuzzy structuring element $1-\nu_B$, according to the original definitions in the fuzzy case [20]. The non-membership function is exactly the dilation of the fuzzy set ν by the fuzzy structuring element μ_B.

Let us now consider the derived dilation, based on the duality principle. Using the standard negation, it is written as:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = (\sup_{y\in\mathscr{S}} t(\mu_B(x-y),\mu(y)), \inf_{y\in\mathscr{S}} T(\nu_B(x-y),\nu(y))). \tag{22}$$

The first term (membership function) is exactly the fuzzy dilation of μ by μ_B, while the second one (non-membership function) is the fuzzy erosion of ν by $1-\nu_B$, according to the original definitions in the fuzzy case [20].

This observation has a nice interpretation, which fits well with intuition. Let (μ,ν) represent a spatial bipolar fuzzy set, where μ is a positive information for the location of an object for instance, and ν a negative information for this location. A bipolar structuring element can represent additional imprecision on the location, or additional possible locations. Dilating (μ,ν) by this bipolar structuring element amounts to dilate μ by μ_B, i.e. the positive region is extended by an amount represented by the positive information encoded in the structuring element. On the contrary, the negative information is eroded by the complement of the negative information encoded in the structuring element. This corresponds well to what would be intuitively expected in such situations. A similar interpretation can be provided for the bipolar fuzzy erosion.

Let us now consider the implication derived from the Lukasiewicz bipolar operators (Equations 14 and 15). The erosion and the dilation are then expressed as:

$$\begin{aligned}
&\forall x\in\mathscr{S}, \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \\
&\inf_{y\in\mathscr{S}}(\min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)),\max(0,\nu(y)+\mu_B(y-x)-1)) = \\
&(\inf_{y\in\mathscr{S}}\min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)), \sup_{y\in\mathscr{S}}\max(0,\nu(y)+\mu_B(y-x)-1)),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\forall x\in\mathscr{S}, \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \\
&(\sup_{y\in\mathscr{S}}\max(0,\mu(y)+\mu_B(x-y)-1), \inf_{y\in\mathscr{S}}\min(1,\nu(y)+1-\mu_B(x-y),\nu_B(x-y)+1-\mu(y)).
\end{aligned} \tag{24}$$

The negative part of the erosion is exactly the fuzzy dilation of ν (negative part of the input bipolar fuzzy set) with the structuring element μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Similarly, the positive part of the dilation is the fuzzy dilation of μ (positive part of the input) by μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Hence for both operators, the "dilation" part (i.e. negative part for the erosion and positive part for the dilation) has always a direct interpretation and is the same as the one obtained using t-representable operators, for t being the Lukasiewicz t-norm.

In the case the structuring element is non bipolar (i.e. $\forall x\in\mathscr{S}, \nu_B(x) = 1-\mu_B(x)$), then the "erosion" part has also a direct interpretation: the positive part of the erosion is the fuzzy erosion of μ by μ_B for the Lukasiewicz t-conorm; the negative part of the dilation is the erosion of ν by μ_B for the Lukasiewicz t-conorm. This case is then equivalent to the one where t-representable operators are used with Lukasiewicz t-norm and t-conorm.

4.5 Ilustrative Example

Let us now illustrate these morphological operations on the simple example shown in Figure 1. Let us assume that an additional information, given as a bipolar structuring element, allows us to reduce the positive part and to extend the negative part of the bipolar fuzzy region. This can be formally expressed as a bipolar fuzzy erosion, applied to the bipolar fuzzy set (μ_L, μ_R), using this structuring element. It corresponds to situations where the initial bipolar fuzzy set was too "permissive" and provided too large possible regions. Figure 5 illustrates the result in the case of a classical structuring element and in the case of a bipolar one. It can be observed that the region corresponding to the positive information has actually been reduced (via a fuzzy erosion), while the region corresponding to the negative part has been extended (via a fuzzy dilation).

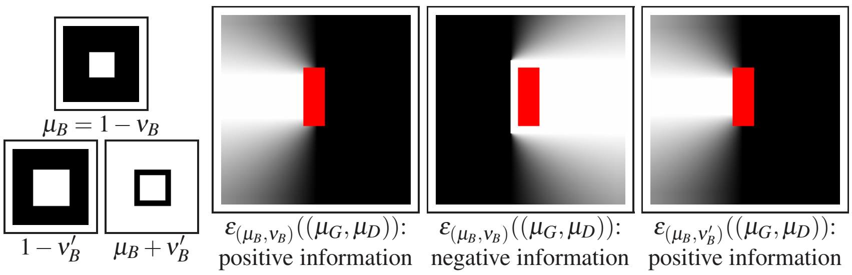

Fig. 5 Illustration of a bipolar fuzzy erosion on the example of Figure 1, using Definition 3 with t-representable operators derived from min and max. A first non bipolar structuring element (μ_B, ν_B) with $\nu_B = 1 - \mu_B$ is used. The results show the reduction of the positive part via an erosion of μ_L with $1 - \nu_B = \mu_B$ and an extension of the negative part via a dilation of μ_L by μ_B. Next, another structuring element, which is truly bipolar, (μ_B, ν'_B) with $\mu_B + \nu'_B \leq 1$ is used. The negative part is the same as in the first case (since μ_B is the same). The positive part undergoes a stronger erosion since $1 - \nu'_B \geq 1 - \nu_B$.

An example of bipolar fuzzy dilation is illustrated in Figure 6 for the bipolar fuzzy set close/far of Figure 2. The dilation corresponds to a situation where the structuring element represents by how much the positive part of the information can be expanded (positive part of the structuring element), for instance because new positions become possible, and by how much the negative part of the information should be reduced (negative part of the structuring element), for instance because it was too severe. These operations allow modifying the semantics attached to the concepts “close” and “far”: in this example, a larger space around the object is considered being close to the object, and the regions that are considered being far from the object are put further away.

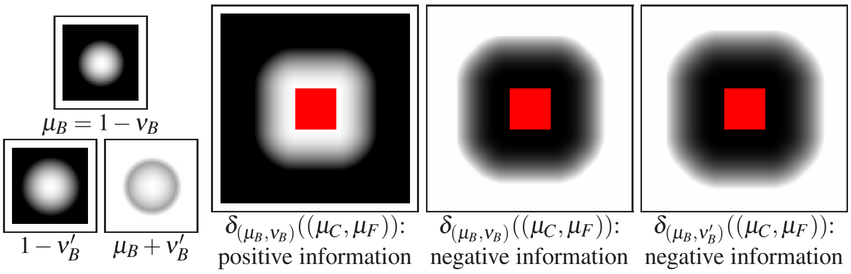

Fig. 6 Illustration of a bipolar fuzzy dilation on the example of Figure 2, using Definition 4 with t-representable operators derived from min and max. Results with a non bipolar fuzzy structuring element (μ_B, ν_B) with $\nu_B = 1 - \mu_B$ show the extension of the positive part via a dilation of μ_C by μ_B and a reduction of the negative part via an erosion of μ_F by $1 - \nu_B = \mu_B$. Another structuring element (μ_B, ν'_B) is used next, which is bipolar: $\mu_B + \nu'_B \leq 1$. The positive part is the same as in the first case (same μ_B). The negative part is more eroded, since $1 - \nu'_B \geq 1 - \nu_B$.

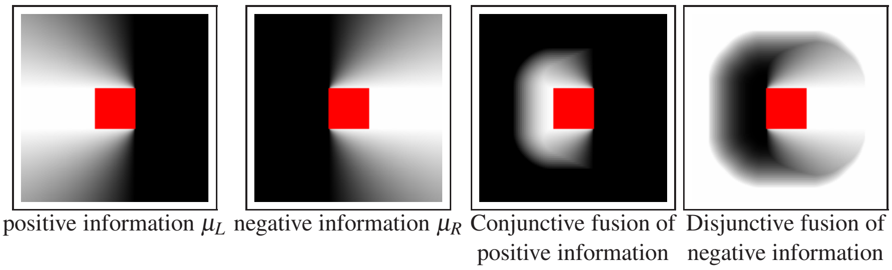

Fig. 7 Fusion of bipolar information on direction (μ_L, μ_R) and on distance $\delta_{(\mu_B,\nu'_B)}((\mu_C,\mu_F))$ of Figure 6

When several pieces of information are available, such as information on direction and information on distance, they can be combined using fusion tools, in order to get a spatial region accounting for all available information. This type of approach has been used to guide the recognition of anatomical structures in images, based on medical knowledge expressed as a set of spatial relations between pairs or triplets of structures (e.g. in an ontology), in the fuzzy case [18, 26, 41]. This idea can be extended to the bipolar case. As an example, a result of fusion of directional and distance information is illustrated in Figure 7. The positive information “to the left” of the reference object (and the negative part “to the right”) is combined with the dilated distance information shown in Figure 6. The positive parts are combined in a conjunctive way (using a min here) and the negative parts in a disjunctive way (as a max here), according to the semantics of the fusion of bipolar information [34]. The meaning of the positive part is then "to the left and close to" and the one of the negative part is "to the right and far from". This example shows how the search space can be reduced by combining spatial relations to reference objects, expressed as bipolar fuzzy sets. This can be considered as an extension to the bipolar case of attention focusing approaches. Further examples will be given in Section 6.

4.6 Derived Operators

Once the two basic morphological operators, erosion and dilation, have been defined on bipolar fuzzy sets, a lot of other operators can be derived in a quite straightforward way. We provide a few examples in this section.

4.6.1 Morphological Gradient

A direct application of erosion and dilation is the morphological gradient, which extracts boundaries of objects by computing the difference between dilation and erosion.

Definition 6. *Let (μ,ν) a bipolar fuzzy set. We denote its dilation by a bipolar fuzzy structuring element by (δ^+,δ^-) and its erosion by $(\varepsilon^+,\varepsilon^-)$. We define the bipolar fuzzy gradient as:*

$$\nabla(\mu,\nu) = (\min(\delta^+,\varepsilon^-), \max(\delta^-,\varepsilon^+)) \tag{25}$$

which is the set difference, expressed as the conjunction between (δ^+,δ^-) and the negation $(\varepsilon^-,\varepsilon^+)$ of $(\varepsilon^+,\varepsilon^-)$.

Proposition 8. *The bipolar fuzzy gradient has the following properties:*

1. *Definition 6 defines a bipolar fuzzy set.*
2. *If the dilation and erosion are defined using t-representable bipolar t-norms and t-conorms, we have:*

$$\nabla(\mu,\nu) = (\min(\delta_{\mu_B}(\mu), \delta_{\mu_B}(\nu)), \max(\varepsilon_{1-\nu_B}(\nu), \varepsilon_{1-\nu_B}(\mu))). \tag{26}$$

   Moreover, if (μ,ν) is not bipolar (i.e. $\nu = 1-\mu$), then the positive part of the gradient is equal to $\min(\delta_{\mu_B}(\mu), 1-\varepsilon_{\mu_B}(\mu))$, which is exactly the morphological gradient in the fuzzy case.

An illustration is displayed in Figure 8. It illustrates both the imprecision (through the fuzziness of the gradient) and the indetermination (through the indetermination between the positive and the negative parts).

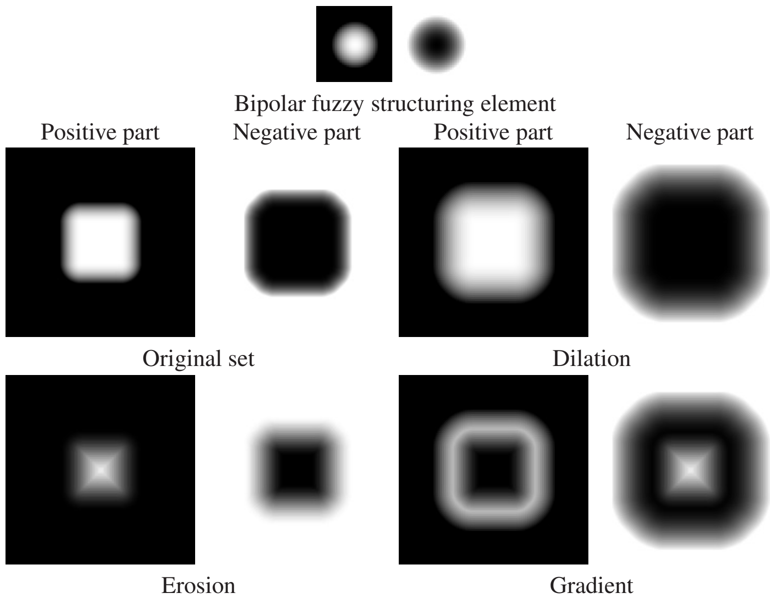

Fig. 8 Gradient using a fuzzy bipolar structuring element and t-representable operators derived from min and max

Another example is shown in Figure 9. The object is here somewhat more complex, and exhibits two different parts, that can be considered as two connected

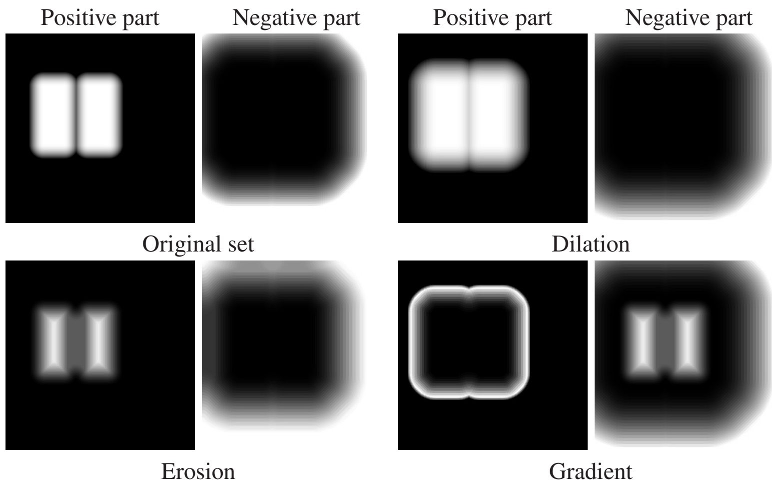

Fig. 9 Gradient using a fuzzy (non bipolar) structuring element ($\nu_B = 1 - \mu_B$ as in Figure 6) on a more complex object

components to some degree. The positive part of the gradient provides a good account of the boundaries of the union of the two components, which amounts to consider that the region between the two components, which has lower membership degrees, actually belongs to the object. The positive part has the expected interpretation as a granted position and extension of the contours. The negative part shows the level of indetermination in the gradient: the gradient could be larger as well, and it could also include the region between the two components.

4.6.2 Conditional Dilation

Another direct application of the basic operators concerns the notion of conditional dilation (respectively conditional erosion) [52]. These operations are very useful in mathematical morphology in order to constrain an operation to provide a result restricted to some region of space. In the digital case, a conditional dilation can be expressed using the intersection of the usual dilation with an elementary structuring element and the conditioning set. This operation is iterated in order to provide the conditional dilation with a larger structuring element. Iterating this operation until convergence leads to the notion of reconstruction. This operation is very useful in cases we have a marker of some objects, and we want to recover the whole objects marked by this marker, and only these objects.

The extension of these types of operations to the bipolar fuzzy case is straightforward: given a bipolar fuzzy marker (μ_M, μ_N), the dilation of (μ_M, μ_N), conditionally to a bipolar fuzzy set (μ, ν) is simply defined as the conjunction of the dilation of (μ_M, μ_N) and (μ, ν). It is easy to show that this defines a bipolar fuzzy set. An example is shown in Figure 10, showing that the conditional dilation of the marker is restricted to only one component (the one including the marker) of the original object (only the positive parts are shown). Iterating further this dilation would provide the whole marked component.

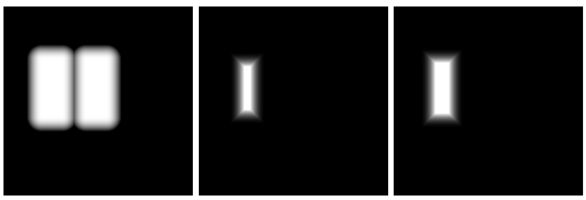

Fig. 10 Conditioning set, marker and conditional dilation (only the positive parts are shown)

4.6.3 Opening, Closing, and Derived Operators

Applying a dilation and then an erosion by the same structuring element defines a closing, while applying first an erosion and then a dilation defines an opening. Thanks to the strong underlying algebraic framework (see [12] for details), opening and closing have all required properties: they are idempotent and increasing (hence they define morphological filters), opening is anti-extensive and closing is extensive (whatever the choice of the structuring element), if Lukasiewicz operators are used (up to a permutation on $[0,1]$) since the adjunction property is required for these properties. The closing of the bipolar fuzzy object shown in Figure 9 is displayed in Figure 11. The small region between the two components in the positive part has been included in this positive part (to some degree) by the closing, which is the expected result.

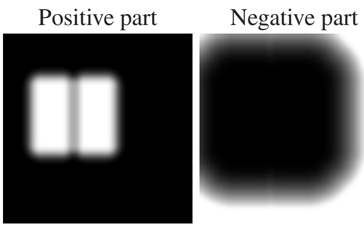

Fig. 11 Bipolar fuzzy closing using Lukasiewicz operators. The fuzzy bipolar structuring element (μ_B, ν'_B) of Figure 6 was used here.

Another example is shown in Figure 12, where some small parts have been introduced in the bipolar fuzzy set. The opening successfully removes these small parts (i.e. small regions with high μ values and small regions with low ν values are removed from the positive part and the negative part, respectively). A typical use of this operation is for situations where the initial bipolar fuzzy set represents possible/forbidden regions for an object. If we have some additional information on the size of the object, so that it is sure that it cannot fit into small parts, then opening can be used to remove possible small places, and to add to the negative part such small regions.

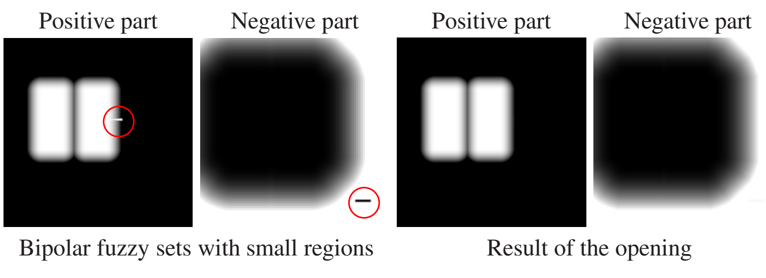

Fig. 12 Bipolar fuzzy opening using Lukasiewicz operators. Circles indicate small regions that are removed by the opening (see text). The bipolar fuzzy structuring element (μ_B, ν'_B) of Figure 6 was used in this example.

From these new operators, a lot of other ones can be derived, extending the classical ones to the bipolar case. For instance, several filters can be deduced from opening and closing, such as alternate sequential filters [52], by applying alternatively opening and closing, with structuring elements of increasing size. Another example is the top-hat transform [52], which allows extracting bright structures having a given approximative shape, using the difference between the original image and the result of an opening using this shape as a structuring element. Such operators can be directly extended to the bipolar case using the proposed framework.

4.7 Distance from a Point to a Bipolar Fuzzy Set

While there is a lot of work on distances and similarity between interval-valued fuzzy sets or between intuitionistic fuzzy sets (see e.g. [55, 57]), none of the existing definitions addresses the question of the distance from a point to a bipolar fuzzy set, nor includes the spatial distance in the proposed definitions. As in the fuzzy case [7], we propose to define the distance from a point to a bipolar fuzzy set using a morphological approach [17]. In the crisp case, the distance from a point x to a set X is equal to n iff x belongs to the dilation of size n of X (the dilation of size 0 being the identity), but not to dilations of smaller size (it is sufficient to test this condition for $n-1$ in the discrete case). The transposition of this property to the bipolar fuzzy case leads to the following definition, using bipolar fuzzy dilations [17].

Definition 7. *The distance from a point x of $\mathscr{S}$ to a bipolar fuzzy set (μ,ν) ($\in \mathscr{B}$) is defined as: $d(x,(\mu,\nu))(0) = (\mu(x),\nu(x))$ and $\forall n \in \mathbb{N}^*, d(x,(\mu,\nu))(n) = \delta^n_{(\mu_B,\nu_B)}(x) \wedge c(\delta^{n-1}_{(\mu_B,\nu_B)}(x))$, where c is a complementation (typically the standard negation $c(a,b) = (b,a)$ is used) and $\delta^n_{(\mu_B,\nu_B)}$ denotes n iterations of the dilation, using the bipolar fuzzy set (μ_B,ν_B) as structuring element.*

In order to clarify the meaning of this definition, let us consider the case where the structuring element is not bipolar, i.e. $\nu_B = 1-\mu_B$. Then the dilation is expressed as: $\delta_{(\mu_B,1-\mu_B)}(\mu,\nu) = (\delta_{\mu_B}(\mu),\varepsilon_{\mu_B}(\nu))$, where $\delta_{\mu_B}(\mu)$ is the fuzzy dilation of μ by μ_B and $\varepsilon_{\mu_B}(\nu)$ is the fuzzy erosion of ν by μ_B. The bipolar degree to which the distance from x to (μ,ν) is equal to n is then written as: $d(x,(\mu,\nu))(n) = (\delta^n_{\mu_B}(\mu) \wedge \varepsilon^{n-1}_{\mu_B}(\nu), \varepsilon^n_{\mu_B}(\nu) \vee \delta^{n-1}_{\mu_B}(\mu))$, i.e. the positive part is the conjunction of the positive part of the dilation of size n (i.e. a dilation of the positive part of the bipolar fuzzy object) and the negative part of the dilation of size $n-1$ (i.e. an erosion of the negative part of the bipolar fuzzy object), and the negative part is the disjunction of the negative part of the dilation of size n (erosion of ν) and the positive part of the dilation of size $n-1$ (dilation of μ).

Proposition 9. *The distance introduced in Definition 7 has the following properties: (i) it is a bipolar fuzzy set on $\mathbb{N}$; (ii) it reduces to the distance from a point to a fuzzy set, as defined in [7], if (μ,ν) and (μ_B,ν_B) are not bipolar (hence the consistency with the classical definition of the distance from a point to a set is achieved as well); (iii) the distance is strictly equal to 0 (i.e. $d(x,(\mu,\nu))(0) = (1,0)$ and*

$\forall n \neq 0, d(x,(\mu,\nu))(n) = (0,1)$) iff $\mu(x) = 1$ and $\nu(x) = 0$, i.e. x completely belongs to the bipolar fuzzy set.

An example is shown in Figure 13. The results are in agreement with what would be intuitively expected. The positive part of the bipolar fuzzy number is put towards higher values of distances when the point is moved to the right of the object. After a number n of dilations, the point completely belongs to the dilated object, and the value to which the distance is equal to n', with $n' > n$, becomes $(0,1)$. Note that the indetermination in the membership or non-membership to the object (which is truly bipolar in this example) is also reflected in the distances.

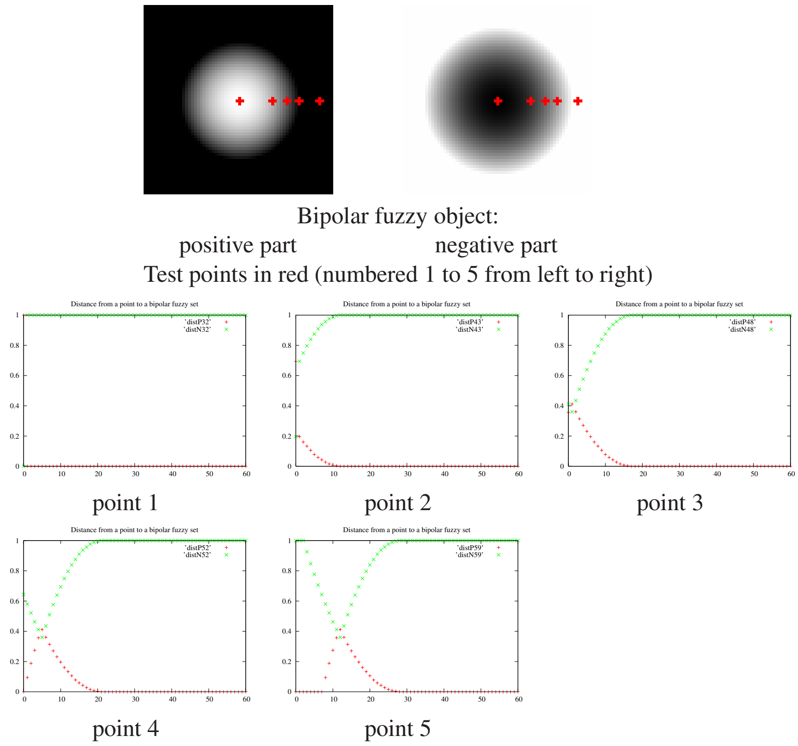

Fig. 13 A bipolar fuzzy set and the distances from 5 different points to it, represented as bipolar fuzzy numbers (positive part in red and negative part in green)

These distances can be easily compared using the extension principle [40, 58], providing a bipolar degree $d_{\leq}$ to which a distance is less than another one. For the examples in Figure 13, we obtain for instance : $d_{\leq}[d(x_1,(\mu,\nu)) \leq d(x_2,(\mu,\nu))] = [0.69, 0.20]$ where x_i denotes the ith point from left to right in the figure. In this case, since x_1 completely belongs to (μ,ν), the degree to which its distance is less than the distance from x_2 to (μ,ν) is equal to $[\sup_a d^+(a), \inf_a d^-(a)]$, where d^+ and

d^- denote the positive and negative parts of $d(x_2, (\mu, \nu))$. As another example, we have $d_{\leq}[d(x_5, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.03, 0.85]$, reflecting that x_5 is clearly not closer to the bipolar fuzzy set (μ, ν) than x_2.

5 Definition of Bipolar Fuzzy Spatial Relations

As mentioned in the introduction, several spatial relations go by pairs of "opposite" relations, such as left/right, above/below, close/far, etc. This is one of the motivations for handling them as bipolar information. Now the question that remains open in the previous sections is: how to define the bipolar fuzzy sets representing these relations in the spatial domain with respect to an object of reference? (i.e. how to construct the representations shown in Figure 1 for instance). Here we assume that the reference object can be crisp or fuzzy, but not bipolar. Our proposal is to rely on our previous work for defining fuzzy spatial representations of spatial relations using dilations with fuzzy structuring elements providing the semantics of the relations [6, 7, 10]. For instance the region to the right of a reference object is defined as the dilation of the reference object with a specific structuring element (see Figure 1 where μ_R and μ_L have been defined using this approach).

Let (μ, ν) be a pair of fuzzy sets in $\mathscr{S}$ representing a pair of "opposite" relations with respect to a reference object. The main problem to be solved is to guarantee that $\forall x \in \mathscr{S}, \mu(x) + \nu(x) \leq 1$. This property may not hold depending on the shape of the reference object: for instance a concavity of an object can be both to the right and to the left of the object, leading to conflicting areas.

In cases the property does not hold directly, we propose three approaches, inspired from [34]:

1. indulgent approach: the positive part is kept unchanged and the negative part is reduced so as to achieve the bipolar constraint. For instance ν can be modified as $\nu'(x) = \min(\nu(x), 1 - \mu(x))$, and (μ, ν') is then bipolar. Note that only the points for which the property is not satisfied are modified. This approach corresponds for instance to cases where the negative part can be interpreted as rules that can be modified in order to achieve consistency with observations.
2. severe approach: the negative part is kept unchanged and the positive part is modified, e.g. $\mu'(x) = \min(\mu(x, 1 - \nu(x))$, so as to have (μ', ν) bipolar. This means that the negative part is privileged in the conflicting areas.
3. tunable compromise: both parts are modified in the conflicting areas, e.g. as:

$$\mu'(x) = \mu(x) - \lambda(\mu(x) + \nu(x) - 1)$$

$$\nu'(x) = \nu(x) - (1 - \lambda)(\mu(x) + \nu(x) - 1)$$

   with $\lambda \in [0, 1]$. Points x for which $\mu(x) + \nu(x) \leq 1$ are not modified. This leads to $\mu'(x) + \nu'(x) = 1$ for the modified points, i.e. the conflict has been replaced by a duality constraint. The first approach is included in this one by taking $\lambda = 0$ and the second one for $\lambda = 1$.

6 Application to Spatial Reasoning

Mathematical morphology provides tools for spatial reasoning at several levels [19]. Its features allow representing objects or object properties, that we do not address here to concentrate rather on tools for representing spatial relations. The notion of structuring element captures the local spatial context, in a fuzzy and bipolar way here, which endows dilation and erosion with a low level spatial reasoning feature, as shown in the interpretation part (Section 4.4). This is then reinforced by the derived operators (opening, closing, gradient, conditional operations...), as introduced for bipolar fuzzy sets in Section 4.6. At a more global level, several spatial relations between spatial entities can be expressed as morphological operations, in particular using dilations [10, 19], leading to large scale spatial reasoning, based for instance on distances [17].

In this section, we illustrate a typical scenario showing the interest of bipolar representations of spatial relations and of morphological operations on these representations for spatial reasoning[15]. Note that this is not a complete application yet, but should only be considered as an illustrative example.

An example of a brain image is shown in Figure 14, with a few labeled structures of interest.

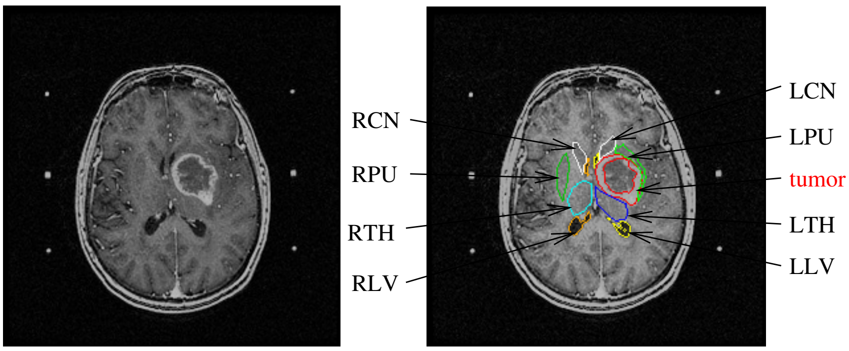

Fig. 14 A slice of a 3D MRI brain image, with a few structures: left and right lateral ventricles (LLV and RLV), caudate nuclei (LCN and RCN), putamen (LPU and RPU) and thalamus (LTH and RTH). A ring-shaped tumor is present in the left hemisphere (the usual "left is right" convention is adopted for the visualization).

Let us first consider the right hemisphere (i.e. the non-pathological one). We consider the problem of defining a region of interest for the RPU, based on a known segmentation of RLV and RTH. An anatomical knowledge base or ontology provides some information about the relative position of these structures [41, 59]:

- directional information: the RPU is exterior (left on the image) of the union of RLV and RTH (positive information) and cannot be interior (negative information);

- distance information: the RPU is quite close to the union of RLV and RTH (positive information) and cannot be very far (negative information).

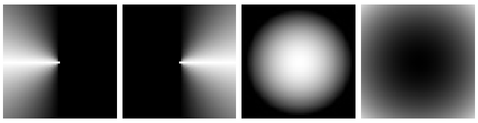

Fig. 15 Fuzzy structuring elements ν_L, ν_R, ν_C and ν_F, defining the semantics of left, right, close and far, respectively

These pieces of information are represented in the image space based on morphological dilations using appropriate structuring elements [10] (representing the semantics of the relations, as displayed in Figure 15) and are illustrated in Figure 16. A bipolar fuzzy set modeling the direction information is defined as:

$$(\mu_{dir}, \nu_{dir}) = (\delta_{\nu_L}(\mathrm{RLV} \cup \mathrm{RTH}), \delta_{\nu_R}(\mathrm{RLV} \cup \mathrm{RTH})),$$

where ν_L and ν_R define the semantics of left and right, respectively. Similarly a bipolar fuzzy set modeling the distance information is defined as:

$$(\mu_{dist}, \nu_{dist}) = (\delta_{\nu_C}(\mathrm{RLV} \cup \mathrm{RTH}), 1 - \delta_{1-\nu_F}(\mathrm{RLV} \cup \mathrm{RTH})),$$

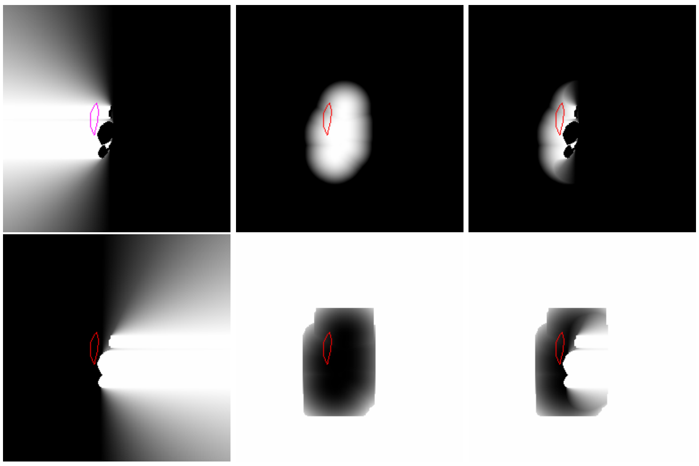

Fig. 16 Bipolar fuzzy representations of spatial relations with respect to RLV and RTH. Top: positive information, bottom: negative information. From left to right: directional relation, distance relation, conjunctive fusion. The contours of the RPU are displayed to show the position of this structure with respect to the region of interest.

where ν_C and ν_F define the semantics of close and far, respectively. The neutral area between positive and negative information allows accounting for potential anatomical variability. The conjunctive fusion of the two types of relations is computed as a conjunction of the positive parts and a disjunction of the negative parts:

$$(\mu_{Fusion}, \nu_{Fusion}) = (\min(\mu_{dir}, \mu_{dist}), \max(\nu_{dir}, \nu_{dist})).$$

As shown in the illustrated example, the RPU is well included in the bipolar fuzzy region of interest which is obtained using this procedure. This region can then be efficiently used to drive a segmentation and recognition technique of the RPU.

Let us now consider the left hemisphere, where a ring-shaped tumor is present. The tumor induces a deformation effect which strongly changes the shape of the normal structures, but also their spatial relations, to a less extent. In particular the LPU is pushed away from the inter-hemispheric plane, and the LTH is pushed towards the posterior part of the brain and compressed. Applying the same procedure as for the right hemisphere does not lead to very satisfactory results in this case (see Figure 18). The default relations are here too strict and the resulting region of interest is not adequate: the LPU only satisfies with low degrees the positive part of the information, while it also slightly overlaps the negative part. In such cases, some relations (in particular metric ones) should be considered with care. This means that they should be more permissive, so as to include a larger area in the possible region, accounting for the deformation induced by the tumor. This can be easily modeled by a bipolar fuzzy dilation of the region of interest with a structuring element (μ_{var}, ν_{var}) (Figure 17), as shown in the last column of Figure 18:

$$(\mu'_{dist}, \nu'_{dist}) = \delta_{(\mu_{var}, \nu_{var})}(\mu_{dist}, \nu_{dist}),$$

where (μ_{dist}, ν_{dist}) is defined as for the other hemisphere. Now the obtained region is larger but includes the correct area. This bipolar dilation amounts to dilate the positive part and to erode the negative part, as explained in Section 4.4.

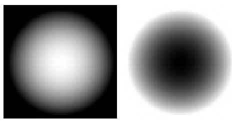

Fig. 17 Bipolar fuzzy structuring element (μ_{var}, ν_{var})

Let us finally consider another example, where we want to use symmetry information to derive a search region for a structure in one hemisphere, based on the segmentation obtained in the other hemisphere. As an illustrative example, we consider the thalamus, and assume that it has been segmented in the non pathological hemisphere (right). Its symmetrical position with respect to the inter-hemispheric plane should provide an adequate search region for the LTH in normal cases. Here this is not the case, because of the deformation induced by the tumor (see Figure 19).

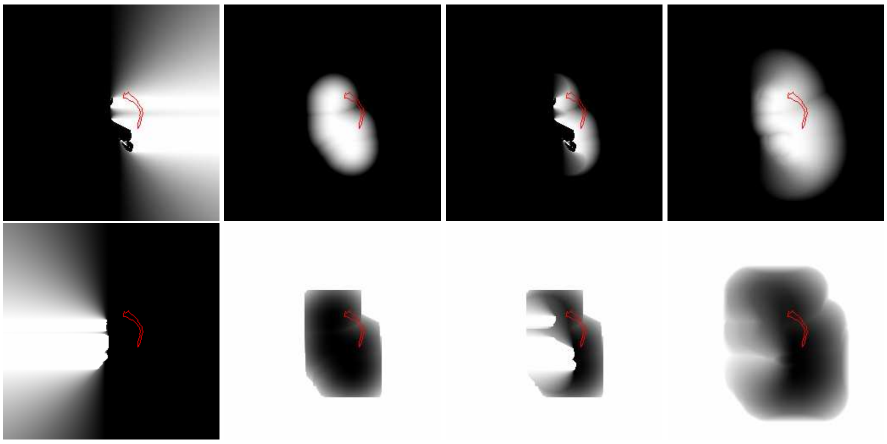

Fig. 18 Bipolar fuzzy representations of spatial relations with respect to LLV and LTH. From left to right: directional relation, distance relation, conjunctive fusion, Bipolar fuzzy dilation. First line: positive parts, second line: negative parts. The contours of the LPU are displayed to show the position of this structure.

Since the brain symmetry is approximate, a small deviation could be expected, but not as large as the one observed here. Here again a bipolar dilation allows defining a proper region, by taking into account both the deformation induced by the tumor and the imprecision in the symmetry.

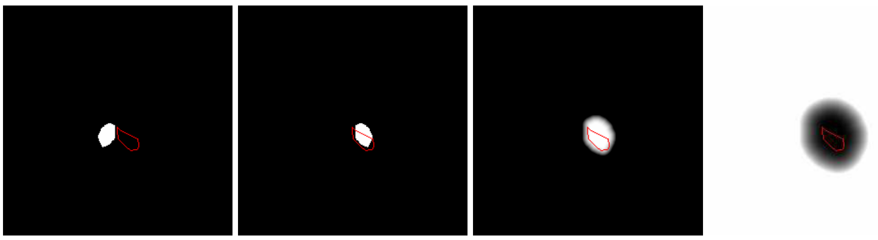

Fig. 19 RTH and its symmetrical, bipolar dilation defining an appropriate search region for the LTH (left: positive part, right: negative part)

7 Conclusion

New concepts on bipolar fuzzy sets are introduced in this paper, in particular geometrical measures, morphological dilations and erosions and derived operators, for which good properties are exhibited and nice interpretations in terms of bipolarity in spatial reasoning can be derived. Further work aims at exploiting these new operations in concrete problems of spatial reasoning, as illustrated in the last part of this paper, in particular for handling the bipolarity nature of some spatial relations. This will require to design a method for evaluating the degree of satisfaction of a bipolar fuzzy relation. Also relations with respect to a bipolar fuzzy set would be an interesting extension of this work.

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