Mathematical Morphology

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A few references

- J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, New-York, 1982.
- J. Serra (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- P. Soille, *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.
- L. Najman, H. Talbot (Eds.), *Mathematical Morphology: From Theory to Applications*, ISTE-Wiley, 2010.

Shape or spatial relations?

Simplifying and selecting relevant information...

Introduction

- Origin: study of porous media
- Principle: study of objects (images) based on:
  - shape, geometry, topology
  - grey levels, colors
  - neighborhood information
- Mathematical bases:
  - set theory
  - topology
  - geometry
  - algebra (lattice theory)
  - probabilities, random closed sets
  - functions
- Main characteristics:
  - non linear
  - non invertible
  - strong properties
  - associated algorithms

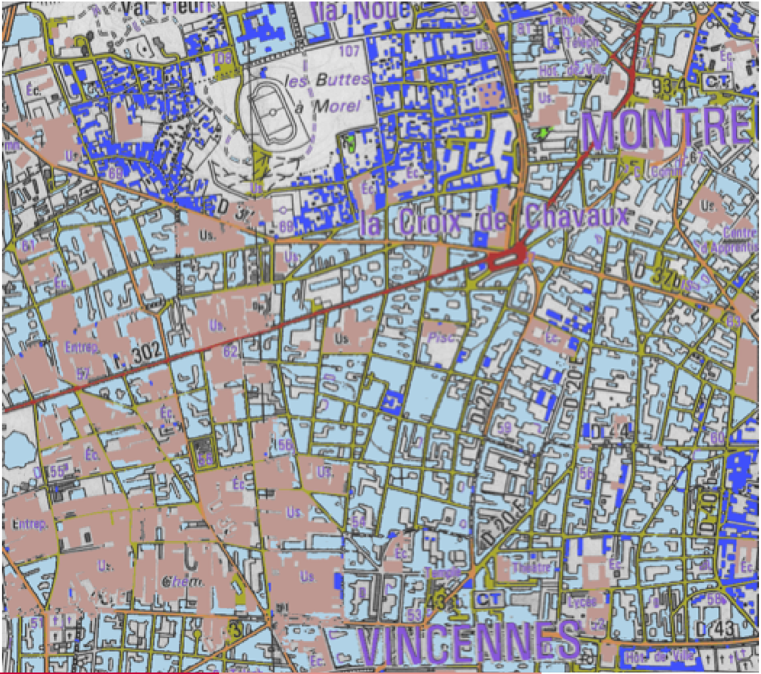
Tools for

- filtering
- segmentation
- measures (distances, granulometry, integral geometry, topology, stochastic processes...)
- texture analysis
- shape recognition
- scene interpretation
- ...

Applications in numerous domains

Example

Val Fleuri
la Noue
les Buttes
à Morel
la Croix de Chavaux
N 302
D 37bis
D 39
D 20
D 20 E
D 44
D 43 E
Entrep.
Théâtre
Lycée
CT

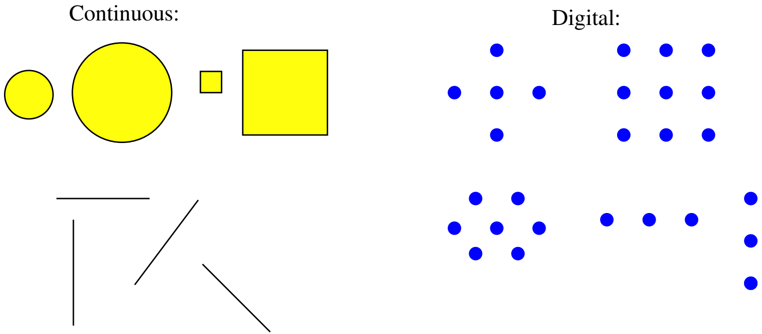

la Noue
les Buttes
à Morel
MONTRE
la Croix de Chavaux
VINCENNES
Entrep.
Théâtre
Lycée
Pisc.
Hôt. de Ville
CT

Four fundamental principles

1. Compatibility with translations
2. Compatibility with scaling
3. Local knowledge
4. Continuity (semi-continuity)

Structuring element

- shape
- size
- origin (not necessarily in B)
- examples:



Binary dilation

Minkowski addition:

$$X \oplus Y = \{x + y \mid x \in X, y \in Y\}$$

Binary dilation:

$$\begin{aligned} D(X, B) &= X \oplus B = \{x + y \mid x \in X, y \in B\} \text{ (or } = X \oplus \check{B} \text{ historically)} \\ &= \bigcup_{x \in X} B_x = \{x \in \mathbb{R}^n \mid \check{B}_x \cap X \neq \emptyset\} \end{aligned}$$

($\check{B}$ = symmetrical of B with respect to space origin, $B_x = x + B$)

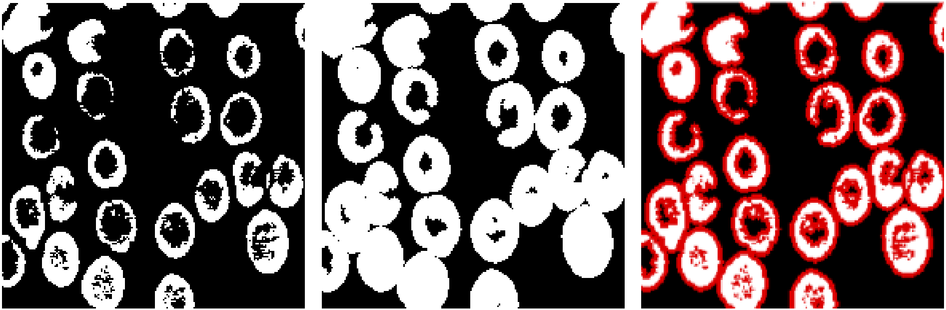
Properties of dilation:

- extensive ($X \subseteq D(X, B)$) iff $O \in B$;
- increasing ($X \subseteq Y \Rightarrow D(X, B) \subseteq D(Y, B)$);
- $B \subseteq B' \Rightarrow D(X, B) \subseteq D(X, B')$;
- commutes with union, not with intersection:

  $$D(X \cup Y, B) = D(X, B) \cup D(Y, B), \quad D(X \cap Y, B) \subseteq D(X, B) \cap D(Y, B);$$

- iterativity property: $D[D(X, B), B'] = D(X, B \oplus B')$.

Example of dilation



Binary erosion

$$\begin{aligned} E(X,B) &= \{x \in \mathbb{R}^n \mid B_x \subseteq X\} \\ &= \{x \in \mathbb{R}^n \mid \forall y \in B, x+y \in X\} = X \ominus \check{B}. \end{aligned}$$

Properties of erosion:

- duality of erosion and dilation with respect to complementation:

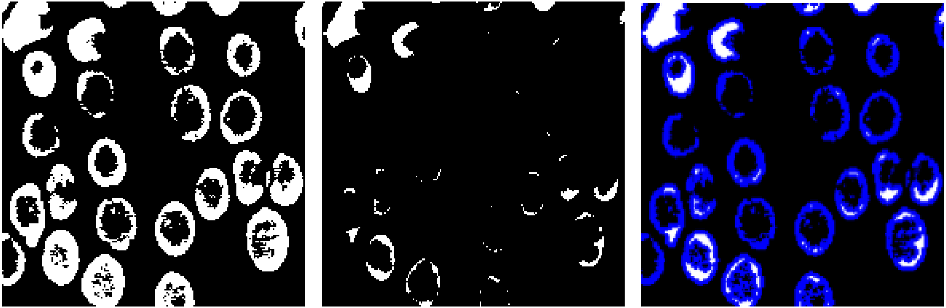
$$E(X,B) = [D(X^C, B)]^C$$

- anti-extensive ($E(X,B) \subseteq X$) iff $O \in B$;
- increasing ($X \subseteq Y \Rightarrow E(X,B) \subseteq E(Y,B)$);
- $B \subseteq B' \Rightarrow E(X,B') \subseteq E(X,B)$;
- commutes with intersection, not with union:

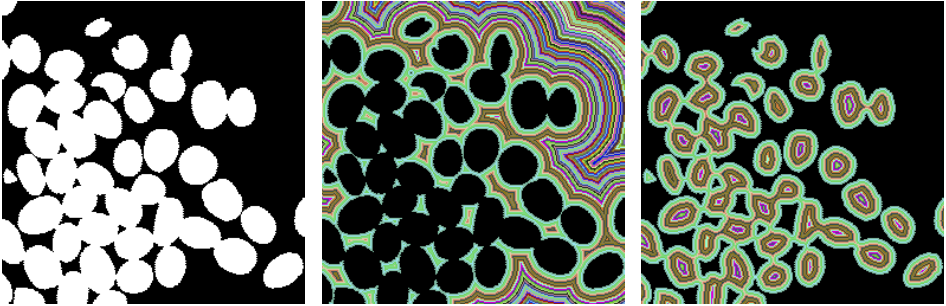
$$E[(X \cap Y), B] = E(X,B) \cap E(Y,B), \quad E[(X \cup Y), B] \supseteq E(X,B) \cup E(Y,B)$$

- iterativity property: $E[E(X,B), B'] = E(X, B \oplus B')$.
- $D[E(X,B), B'] \subseteq E[D(X,B'), B]$.

Example of erosion



Links with distances



Binary opening

$$\boxed{X_B = D[E(X, B), B]}$$

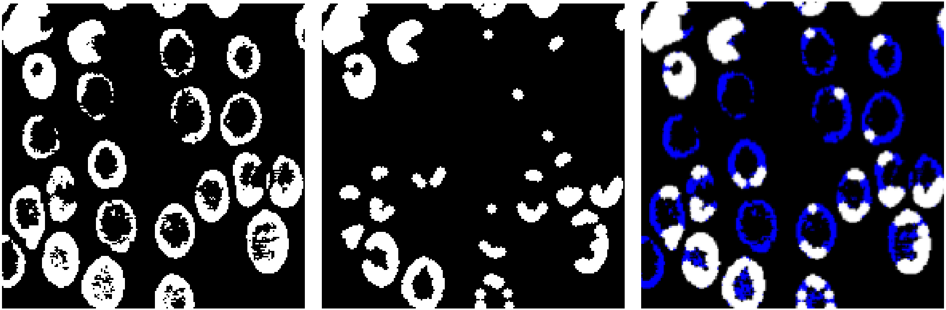
Properties of opening:

- anti-extensive ($X \supseteq X_B$);
- increasing ($X \subseteq Y \Rightarrow X_B \subseteq Y_B$);
- idempotent ($(X_B)_B = X_B$).

$\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X_{B'} \subseteq X_B$;
- $(X_n)_{n'} = (X_{n'})_n = X_{\max(n,n')}$ (with X_n opening with a structuring element of size n).

Example of opening



Binary closing

$$\boxed{X^B = E[D(X,B),B]}$$

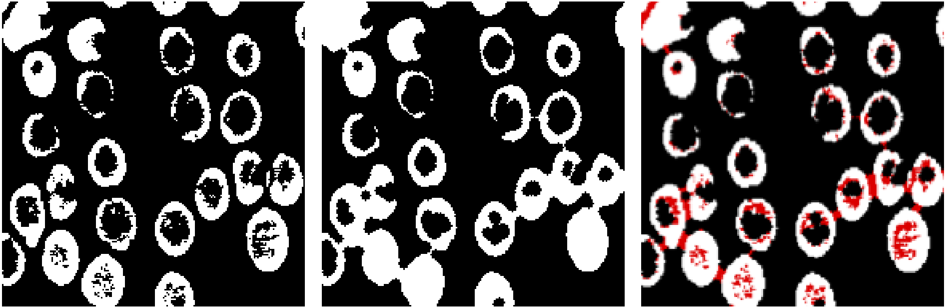
Properties of closing:

- extensive ($X \subseteq X^B$);
- increasing ($X \subseteq Y \Rightarrow X^B \subseteq Y^B$);
- idempotent ($(X^B)^B = X^B$).

$\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X^B \subseteq X^{B'}$;
- $(X^n)^{n'} = (X^{n'})^n = X^{\max(n,n')}$;
- $X^B = [(X^C)_B]^C$.

Example of closing



Digital case

- choice of the digital grid (both for the image and the structuring element)
- translations on the grid
- same properties

From sets to functions

- subgraph of a function on $\mathbb{R}^n$ = subset of $\mathbb{R}^{n+1}$
- cuts of a function = sets

$$f_\lambda = \{x | f(x) \geq \lambda\}$$

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

- functional equivalents of set operations:

$$\begin{array}{ccc} \cup & \rightarrow & \sup / \vee \\ \cap & \rightarrow & \inf / \wedge \\ \subseteq & \rightarrow & \leq \\ \supseteq & \rightarrow & \geq \end{array}$$

Dilation of a function by a flat structuring element

$$\boxed{\forall x \in \mathbb{R}^n, \quad D(f, B)(x) = \sup\{f(y) \mid y \in B_x\}}$$

Properties of functional dilation:

- extensivity iff $O \in B$;
- increasingness;
- $D(f \vee g, B) = D(f, B) \vee D(g, B)$;
- $D(f \wedge g, B) \leq D(f, B) \wedge D(g, B)$;
- iterativity property.

It holds:

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

Example of functional dilation

Erosion of a function

$$\boxed{\forall x \in \mathbb{R}^n, \quad E(f, B)(x) = \inf\{f(y) \mid y \in B_x\}}$$

Properties of functional erosion:

- functional dilation and erosion are dual operators;
- anti-extensivity iff $O \in B$;
- increasingness;
- $E(f \vee g, B) \geq E(f, B) \vee E(g, B)$;
- $E(f \wedge g, B) = E(f, B) \wedge E(g, B)$;
- iterativity property.

Example of functional erosion

Functional opening

$$\boxed{f_B = D[E(f, B), B]}$$

Properties of functional opening:

- anti-extensive;
- increasing;
- idempotent.

$\Rightarrow$ morphological filter

Example of functional opening

Functional closing

$$\boxed{f^B = E[D(f,B),B]}$$

Properties of functional closing:

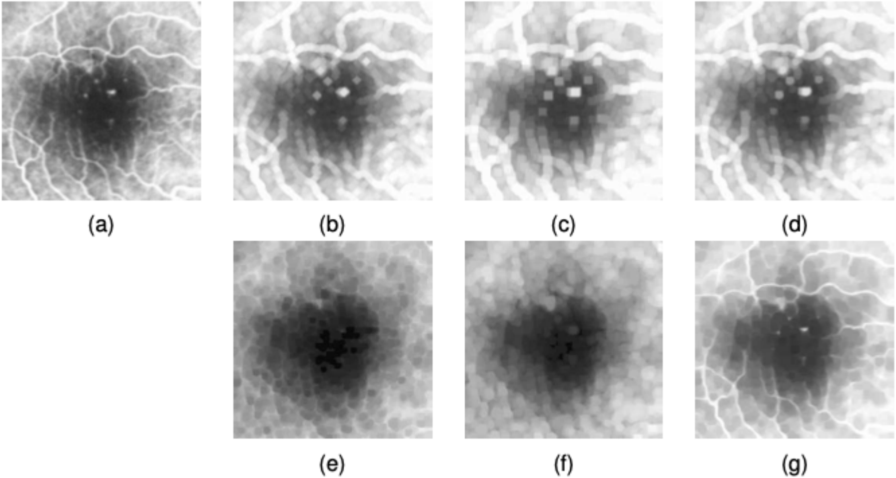
- extensive;
- increasing;
- idempotent.

$\Rightarrow$ morphological filter

- duality between opening and closing

Example of functional closing

Example



(a) Original image. Dilation of size 4 using (b) 4-connectivity structuring element, (c) 8-connectivity structuring element, (d) a discrete approximation of a disk. (e) Erosion of size 4 using 4-connectivity structuring element. (f) Opening of size 4. (g) Closing of size 4.

Structuring functions

Dilation:

$$D(f,g)(x) = \sup_y \{f(y) + g(y-x)\}$$

Erosion:

$$E(f,g)(x) = \inf_y \{f(y) - g(y-x)\}$$

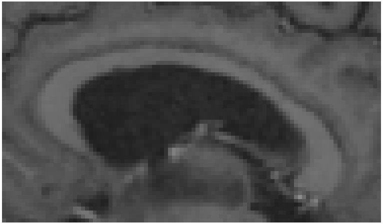
Flat structuring element:

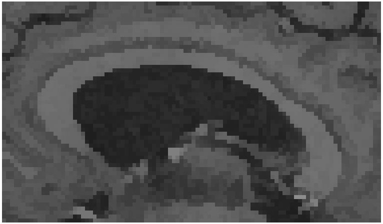
$$g(x) = \begin{cases} 0 & \text{on a compact set } B \\ -\infty & \text{elsewhere} \end{cases}$$

Contrast enhancement

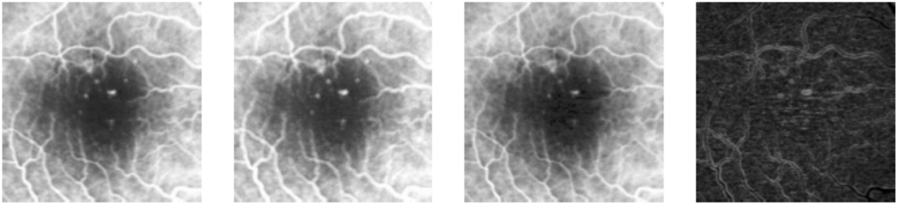
Contrast enhancement: ES 15, $\alpha = \beta = 0.2$, $\alpha = \beta = 0.3$, $\alpha = \beta = 0.5$

Contrast enhancement: ES 30, $\alpha = \beta = 0.2$, $\alpha = \beta = 0.3$, $\alpha = \beta = 0.5$



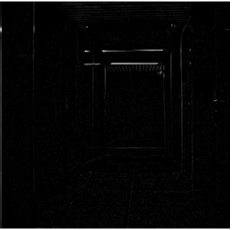


Morphological gradient: $D_B(x) - E_B(x)$



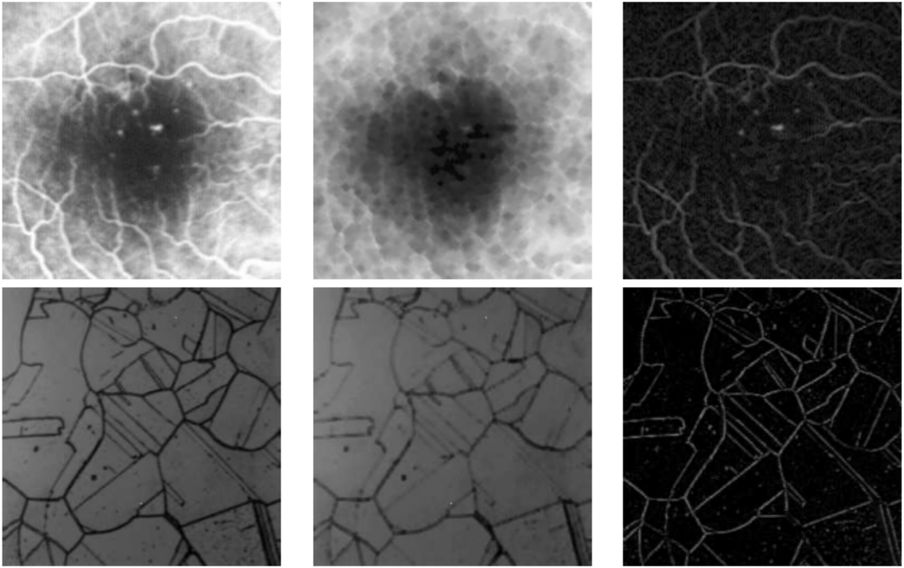
An application of opening: top-hat transform

$$\boxed{f - f_B}$$



An application of opening: top-hat transform

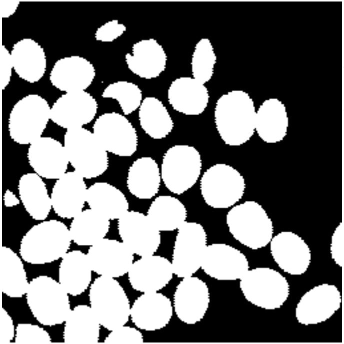
$$\boxed{f - f_B}$$

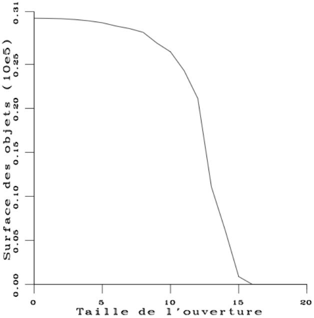


Granulometry

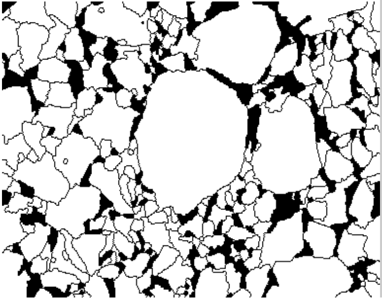
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \ \ \phi_\lambda(X) \subseteq X$ (ϕ_λ anti-extensive);
- $\forall (X, Y) \in \mathcal{A}^2, \forall \lambda > 0, \ \ X \subseteq Y \Rightarrow \phi_\lambda(X) \subseteq \phi_\lambda(Y)$ (ϕ_λ increasing);
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \forall \mu > 0 \ \ \lambda \geq \mu \Rightarrow \phi_\lambda(X) \subseteq \phi_\mu(X)$ (ϕ_λ decreasing with respect to the parameter);
- $\forall \lambda > 0, \forall \mu > 0, \ \ \phi_\lambda \circ \phi_\mu = \phi_\mu \circ \phi_\lambda = \phi_{\max(\lambda,\mu)}$.

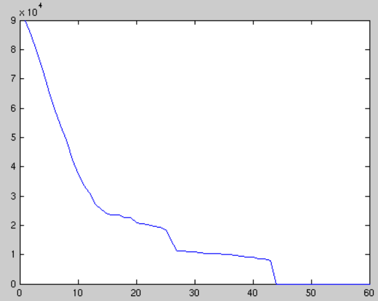
(ϕ_λ) is a granulometry iff ϕ_λ is an opening for each λ and the class of subsets $\mathcal{A}$ which are invariant under ϕ_λ is included in the class of subsets which are invariant under ϕ_μ for $\lambda \geq \mu$





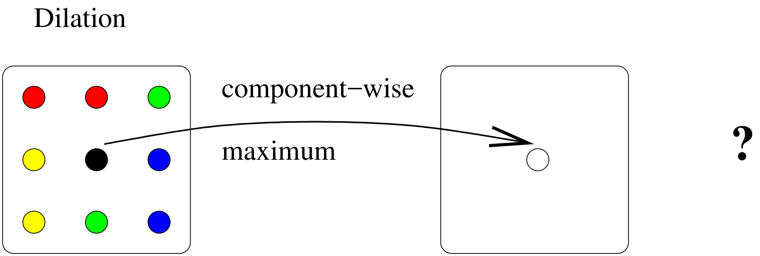
Surface des objets (10e5)
0.00
0.05
0.10
0.15
0.20
0.25
0.31
0
5
10
15
20
Taille de l'ouverture




× 10^4
9
8
7
6
5
4
3
2
1
0
0
10
20
30
40
50
60

Vectorial functions (e.g. color images)

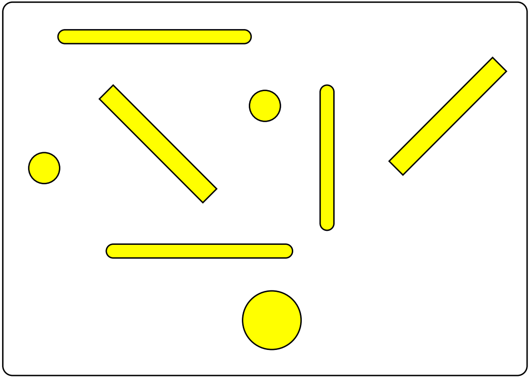
- Main difficulty: choice of an ordering
- component-wise max (or min): no good properties



Choice of the structuring element

- depends on what one wants suppress / keep
- shape
- size

Example: opening by disks or segments?



Choice of the structuring element

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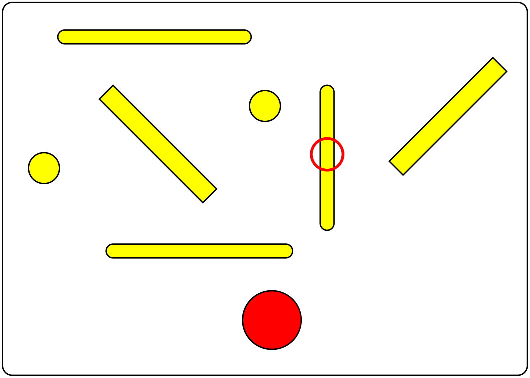
Example: opening by disks or segments?



Choice of the structuring element

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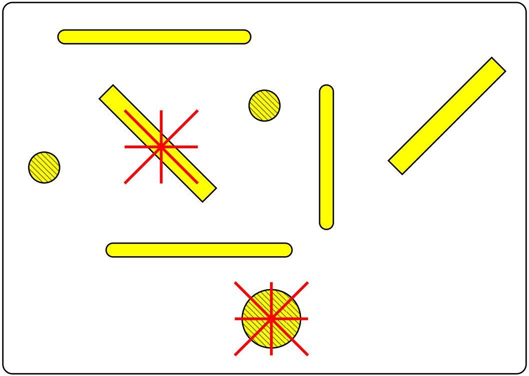
Example: opening by disks or segments?



Rq: a union of openings is an opening

Mathematical bases of mathematical morphology

- Set theory
  - relations ($\subseteq$, $\cap$, $\cup$...)
  - structuring element
- Topology
  - hit-or-miss topology (Fell's topology)
  - myopic topology
  - Hausdorff distance
- Lattice theory
  - adjunctions
  - algebraic operations
- Probability theory
  - $P(A \cap K \neq \emptyset)$
  - random closed sets

Hit-or-miss topology

- topology on closed subsets
- generated by $\mathcal{F}^K$ and $\mathcal{F}_G$ (K compact and G open):

$$\mathcal{F}^K = \{F \in \mathcal{F}, F \cap K = \emptyset\}$$

$$\mathcal{F}_G = \{F \in \mathcal{F}, F \cap G \neq \emptyset\}$$

- convergence in $\mathcal{F}$: $(F_n)_{n \in \mathbb{N}}$ converges towards $F \in \mathcal{F}$ if:

$$\begin{cases} \forall G \in \mathcal{G}, G \cap F \neq \emptyset, \exists N, \forall n \geq N, \quad G \cap F_n \neq \emptyset \\ \forall K \in \mathcal{K}, K \cap F = \emptyset, \exists N', \forall n \geq N', \; K \cap F_n = \emptyset \end{cases}$$

Union is continuous from $\mathcal{F} \times \mathcal{F}$ in $\mathcal{F}$ but intersection is not

$\Downarrow$

semi-continuity

Semi-continuity

$f : \Omega \to \mathcal{F}$

- f upper semi-continuous (u.s.c.) if $\forall \omega \in \Omega$ and $\forall (\omega_n)_{n \in \mathbb{N}} \in \Omega$ converging towards ω :

$$\overline{lim} f(\omega_n) \subseteq f(\omega)$$

- f lower semi-continuous (l.s.c.) if:

$$\underline{lim} f(\omega_n) \supseteq f(\omega)$$

$\overline{lim} / \underline{lim} = \cup / \cap$ of adherence points

f continuous iff f l.s.c. and u.s.c.

Intersection is u.s.c.

Properties of morphological operations

- the dilation of a closed set by a compact set is continuous
- the dilation of a compact set by a compact set is continuous
- $(F, K) \mapsto E(F, K)$ u.s.c.
- $(K', K) \mapsto E(K', K)$ u.s.c.
- $(F, K) \mapsto F_K$ u.s.c.
- $(K', K) \mapsto K'_K$ u.s.c.
- $(F, K) \mapsto F^K$ u.s.c.
- $(K', K) \mapsto K'^K$ u.s.c.

Myopic topology

- generated by:

$$\mathcal{K}_G^F = \{K \in \mathcal{K}, K \cap F = \emptyset, K \cap G \neq \emptyset\}$$

  $(F \in \mathcal{F},\ G \in \mathcal{G})$
- finer than the topology induced on $\mathcal{K}$ by the hit-or-miss topology
- equivalent on $\mathcal{K} \setminus \emptyset$ to the topology induced by the Hausdorff distance

$$\delta(K, K') = \max\{\sup_{x \in K} d(x, K'), \sup_{x' \in K'} d(x', K)\}$$

Rq: $\delta(K, K') = \inf\{\varepsilon,\ K \subseteq D(K', B^\varepsilon), K' \subseteq D(K, B^\varepsilon)\}$

Algebraic framework: complete lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ ordering) such that $\forall (x,y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound
- $\Rightarrow$ contains a smallest element 0 and a largest element I:

$$0 = \bigwedge \mathcal{T} = \bigvee \emptyset \text{ et } I = \bigvee \mathcal{T} = \bigwedge \emptyset$$

- Examples of complete lattices:
  - $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive):

$$\forall x, \exists x^C, x \wedge x^C = 0 \text{ and } x \vee x^C = I$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

  - $(\mathcal{F}(\mathbb{R}^d), \subseteq)$
  - functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the ordering $\leq$:

$$f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, \;\; f(x) \leq g(x)$$

  - partitions

Semi-continuity of functions

- u.s.c. :
$$\forall t > f(x), \exists V(x), \forall y \in V(x), \quad t > f(y)$$
($V(x)$ neighborhood of x in $\mathbb{R}^n$)
- l.s.c. :
$$\forall t < f(x), \exists V(x), \forall y \in V(x), t < f(y)$$

- a function is u.s.c. iff its sub-graph is closed
- topology on the space of u.s.c. functions = topology induced by the hit-or-miss topology on $\mathcal{F}(\mathbb{R}^n \times \overline{\mathbb{R}})$
- the set of u.s.c. functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ is a complete lattice for $\leq$:
$$f \leq g \Leftrightarrow SG(f) \subseteq SG(g)$$

Algebraic dilation and erosion

Complete lattice $(\mathcal{T}, \leq)$

Algebraic dilation:

$$\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee_i \delta(x_i)$$

Algebraic erosion:

$$\forall (x_i) \in \mathcal{T},\ \varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i)$$

Properties:

- $\delta(0) = 0$ (in $\mathcal{P}(E), 0 = \emptyset$)
- $\varepsilon(I) = I$ (in $\mathcal{P}(E), I = E$)
- δ increasing
- ε increasing
- in $\mathcal{P}(\mathbb{R}^n)$, $\delta(X) = \cup_{x \in X} \delta(\{x\})$

Adjunctions

(ε, δ) adjunction on $(\mathcal{T}, \leq)$:

$$\forall (x, y),\ \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0$ and $\varepsilon(I) = I$
- (ε, δ) adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\delta =$ algebraic dilation
- δ increasing $=$ algebraic dilation iff $\exists \varepsilon$ such that (ε, δ) is an adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\varepsilon(x) = \bigvee \{y \in \mathcal{T},\ \delta(y) \leq x\}$
- ε increasing $=$ algebraic erosion iff $\exists \delta$ such that (ε, δ) is an adjunction $\Rightarrow \delta =$ algebraic dilation and $\delta(x) = \bigwedge \{y \in \mathcal{T},\ \varepsilon(y) \geq x\}$
- $\varepsilon\delta \geq Id$
- $\delta\varepsilon \leq Id$
- $\varepsilon\delta\varepsilon = \varepsilon$
- $\delta\varepsilon\delta = \delta$
- $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$

Links with morphological operators

- On the lattice of the subsets of $\mathbb{R}^n$ or $\mathbb{Z}^n$, with inclusion:

$$\delta(X) = \cup_{x \in X} \delta(\{x\})$$

- + invariance under translation $\Rightarrow \exists B,\ \delta(X) = D(X, B)$
- Same result on the lattice of functions.
- Similar results for erosion.

Algebraic opening and closing

- Algebraic opening: γ increasing, idempotent and anti-extensive
- Algebraic closing: φ increasing, idempotent and extensive
- Examples: $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ with (ε, δ) = adjunction
- Invariance domain: $Inv(\varphi) = \{x \in \mathcal{T},\ \varphi(x) = x\}$
- γ opening $\Rightarrow \gamma(x) = \bigvee\{y \in Inv(\gamma),\ y \leq x\}$
- φ closing $\Rightarrow \varphi(x) = \bigwedge\{y \in Inv(\varphi),\ x \leq y\}$
- (γ_i) openings $\Rightarrow \bigvee_i \gamma_i$ opening
- (φ_i) closings $\Rightarrow \bigwedge_i \varphi_i$ closing
- γ_1 and γ_2 openings $\Rightarrow$ equivalence between:
  1. $\gamma_1 \leq \gamma_2$
  2. $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
  3. $Inv(\gamma_1) \subseteq Inv(\gamma_2)$
- φ_1 and φ_2 closings $\Rightarrow$ equivalence between:
  1. $\varphi_2 \leq \varphi_1$
  2. $\varphi_1\varphi_2 = \varphi_2\varphi_1 = \varphi_1$
  3. $Inv(\varphi_1) \subseteq Inv(\varphi_2)$

Filter = increasing and idempotent operator

Examples

- openings γ and $\bigvee_i \gamma_i$ (anti-extensive filters)
- closings φ and $\bigwedge_i \varphi_i$ (extensive filters)

Theorem on filter composition φ and ψ such that $\varphi \geq \psi$:

- $\varphi \geq \varphi\psi\varphi \geq \varphi\psi \vee \psi\varphi \geq \varphi\psi \wedge \psi\varphi \geq \psi\varphi\psi \geq \psi$
- $\varphi\psi$, $\psi\varphi$, $\varphi\psi\varphi$ and $\psi\varphi\psi$ are filters
- $Inv(\varphi\psi\varphi) = Inv(\varphi\psi)$ and $Inv(\psi\varphi\psi) = Inv(\psi\varphi)$
- $\varphi\psi\varphi$ is the smallest filter which is largest than $\varphi\psi \vee \psi\varphi$

Example: alternate sequential filters

- openings γ_i and closings φ_i such that:

$$i \leq j \Rightarrow \gamma_j \leq \gamma_i \leq Id \leq \varphi_i \leq \varphi_j$$

- Theorem on filter composition $\Rightarrow m_i = \gamma_i\varphi_i$, $n_i = \varphi_i\gamma_i$, $r_i = \varphi_i\gamma_i\varphi_i$ and $s_i = \gamma_i\varphi_i\gamma_i$ are filters
- Alternate sequential filters:

$$\begin{aligned} M_i &= m_i m_{i-1} ... m_2 m_1 \\ N_i &= n_i n_{i-1} ... n_2 n_1 \\ R_i &= r_i r_{i-1} ... r_2 r_1 \\ S_i &= s_i s_{i-1} ... s_2 s_1 \end{aligned}$$

- Property: $i \leq j \Rightarrow M_j M_i = M_j,\ N_j N_i = N_j, ...$

Morphological alternate sequential filters

$$(\dots(((f_{B_1})^{B_1})_{B_2})^{B_2})\dots_{B_n})^{B_n}$$

Morphological alternate sequential filters

$$(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}$$

Comparison of filters on an image with Gaussian noise

Original image

Gaussian noise (variance 20)

3×3 mean

7×7 mean

Gaussian filter with variance 0.75

Gaussian filter with variance 4.08

Noisy image

Nagao filter

3×3 median

7×7 median

Noisy image

Alternate sequential filter 1

Alternate sequential filter 2

Alternate sequential filter 3

Original image

Gaussian noise (variance 120)

3×3 mean

7×7 mean

Gaussian filter with variance 0.75

Gaussian filter with variance 4.08

Noisy image

Nagao filter

3×3 median

7×7 median

Noisy image

Alternate sequential filter 1

Alternate sequential filter 2

Alternate sequential filter 3

Comparison of filters on an image with impulse noise

Original image

Impulse noise (intensity 2%)

3×3 mean

7×7 mean

Gaussian filter with variance 0.75

Gaussian filter with variance 4.08

Noisy image

Nagao filter

3×3 median

7×7 median

Noisy image

Alternate sequential filter 1

Alternate sequential filter 2

Alternate sequential filter 3

Original image

Impulse noise (intensity 10%)

3×3 mean

7×7 mean

Gaussian filter with variance 0.75

Gaussian filter with variance 4.08

Noisy image

Nagao filter

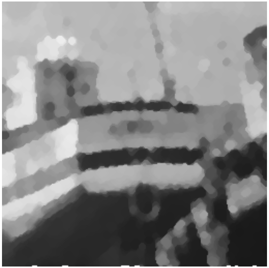
3×3 median

7×7 median

Noisy image

Alternate sequential filter 1

Alternate sequential filter 2



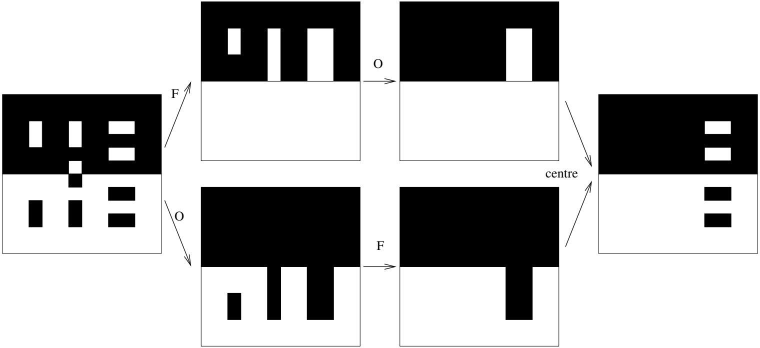
Alternate sequential filter 3

Auto-dual filters

- Operators which are independent of the local contrast, acting similarly on bright and dark areas.
- Example: morphological center

$$Median[f(x), \psi_1(f)(x), \psi_2(f)(x)]$$

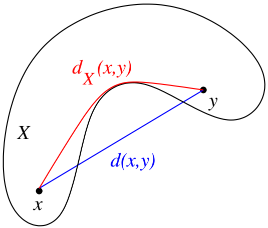
- More generally, for operators $\{\psi_1, \psi_2, ...\psi_n\}$: $(Id \vee \wedge_i \psi_i) \wedge \vee_i \psi_i$
- For instance $\psi_1(f) = \gamma\varphi(f) = (f^B)_B$, $\psi_2 = \varphi\gamma(f) = (f_B)^B$


F
O
O
F
centre

Geodesic operators

Geodesic distance, conditional to X: d_X

- if X is closed, there exits a geodesic arc for any pair of points of X
- unique if X is simply connected
- X convex $\Leftrightarrow$ $d_X = d$



Geodesic ball: $B_X(x, r) = \{y \in X \mid d_X(x, y) \leq r\}$
Rq: $B_X(x, r) \subseteq B(x, r)$

Geodesic dilation:

$$D_X(Y, B_r) = \{x \in \mathbb{R}^n \mid B_X(x, r) \cap Y \neq \emptyset\} = \{x \in \mathbb{R}^n \mid d_X(x, Y) \leq r\}$$

Geodesic erosion:

$$E_X(Y, B_r) = \{x \in \mathbb{R}^n \mid B_X(x, r) \subseteq Y\} = X \setminus D_X(X \setminus Y, B_r)$$

Geodesic opening and closing: by composition

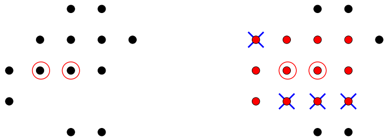
Properties and reconstruction

Properties:

- similar as in the Euclidean case
- $D_X(Y, B_r) \subseteq D(Y, B_r)$
- $D_X(Y, B_r) = \cap_{n=1}^{\infty}[(Y \oplus \frac{r}{n}B) \cap X]^n$

Digital case:

$$D_X(Y, B_r) = [D(Y, B_1) \cap X]^r$$

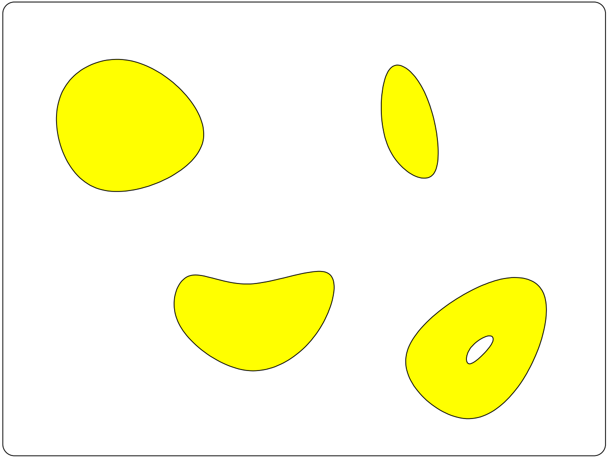


Reconstruction:

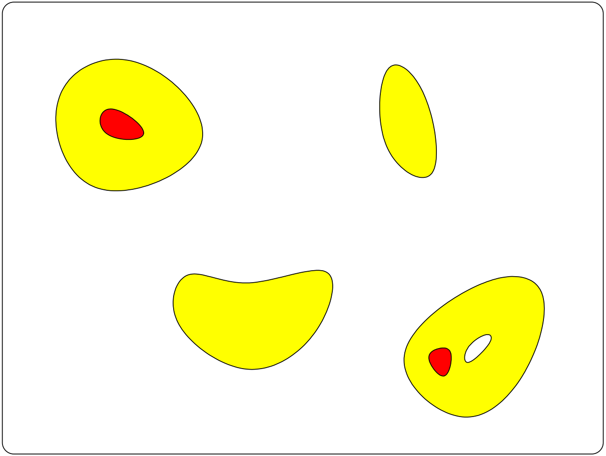
$$[D(Y, B_1) \cap X]^{\infty} = D_X^{\infty}(Y)$$

= connected components of X which intersect Y

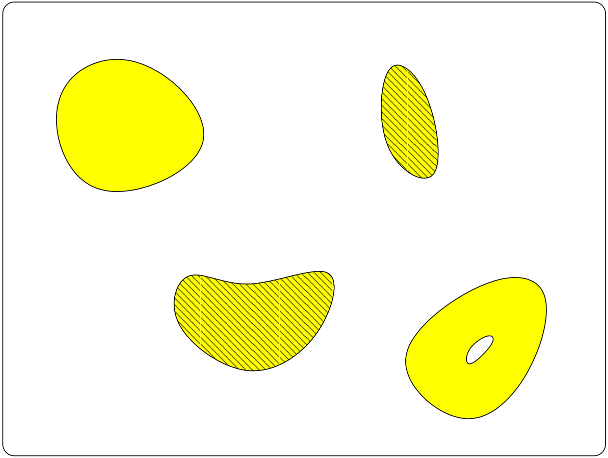
Binary reconstruction: example



Binary reconstruction: example



Binary reconstruction: example

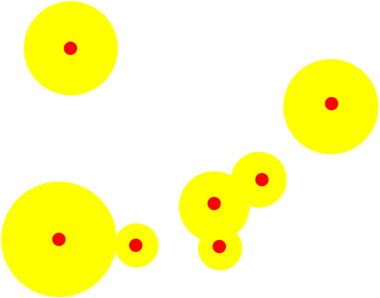


Ultimate erosion

$$EU(X) = \cup_n \{E(X, B_n) \setminus R[E(X, B_{n+1}); E(X, B_n)]\}$$

- $E(X, B_n)$: erosion of X by a structuring element of size n
- $R[Y; Z]$: connected components of Z having a non-empty intersection with Y

$=$ set of regional maxima of the distance function $d(x, X^C)$.



Geodesic operators on functions

$$X_1 \subseteq X_2 \text{ and } Y_1 \subseteq Y_2 \;\Rightarrow\; D_{X_1}(Y_1, B_r) \subseteq D_{X_2}(Y_1, B_r) \subseteq D_{X_2}(Y_2, B_r)$$

$\Rightarrow$ Extension to functions, for $f \leq g$, cut by cut:

$$[D_g(f, B_r)]_\lambda = D_{g_\lambda}(f_\lambda, B_r)$$

(with $f_\lambda = \{x,\ f(x) \geq \lambda\}$)
Digital case:

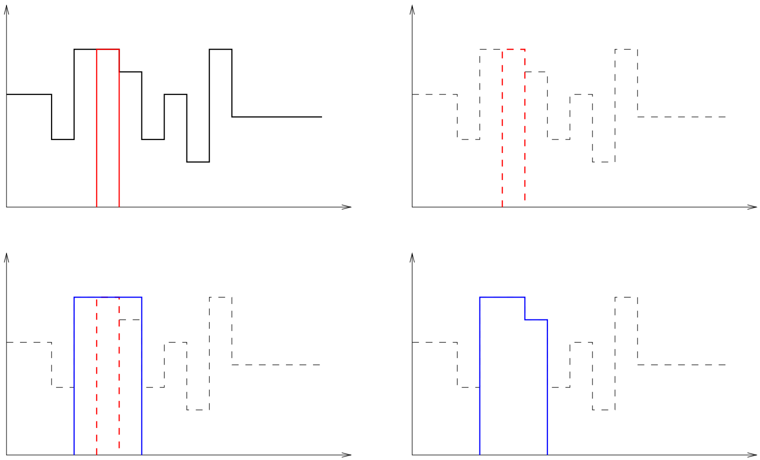
$$D_g(f, B_r) = [D(f, B_1) \wedge g]^r$$

$$E_g(f, B_r) = [E(f, B_1) \vee g]^r$$

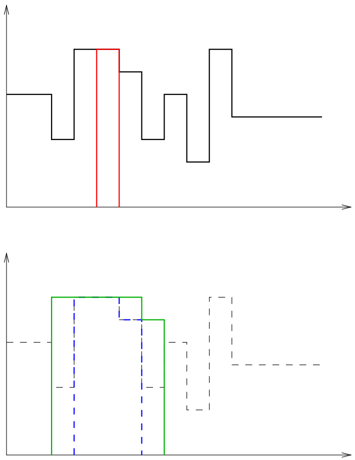
Numerical reconstruction of f (marker function) in g:

- by dilation $D_g(f, B_\infty) = D_g^\infty(f)$: opening
- by erosion $E_g(f, B_\infty)$: closing
- opening by reconstruction: $D_f^\infty(f_B)$ (flat areas whose contours are some contours of the original image $\Rightarrow$ compression)

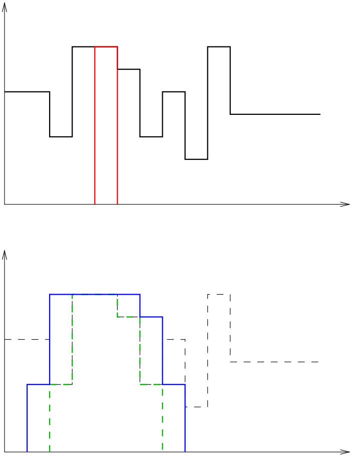
Numerical reconstruction: example



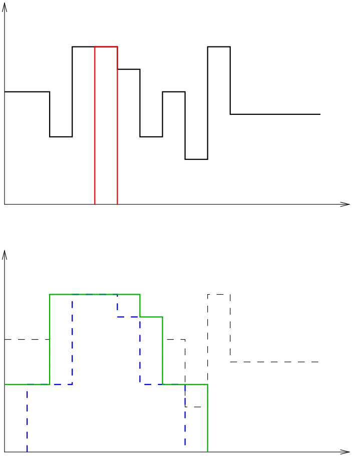
Numerical reconstruction: example



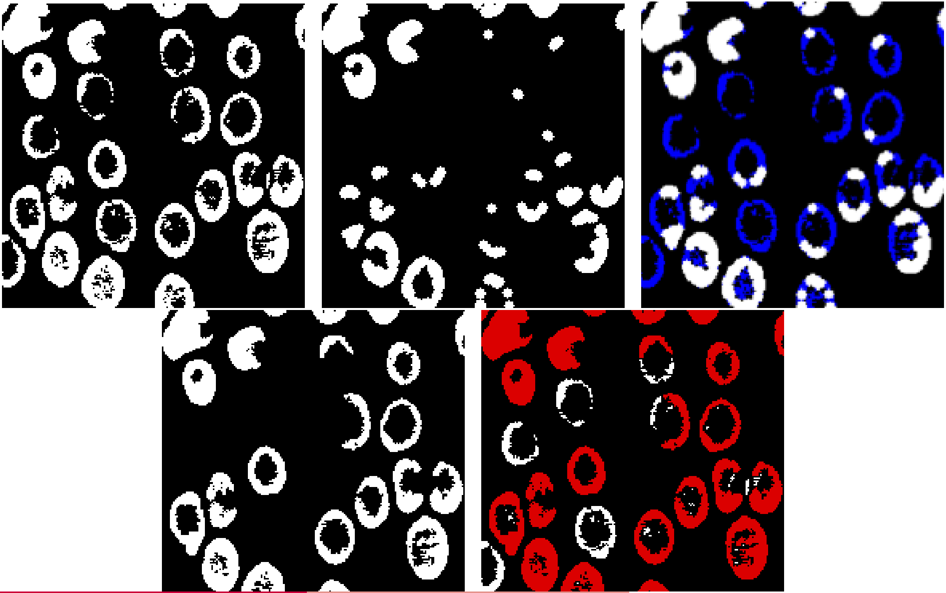
Numerical reconstruction: example

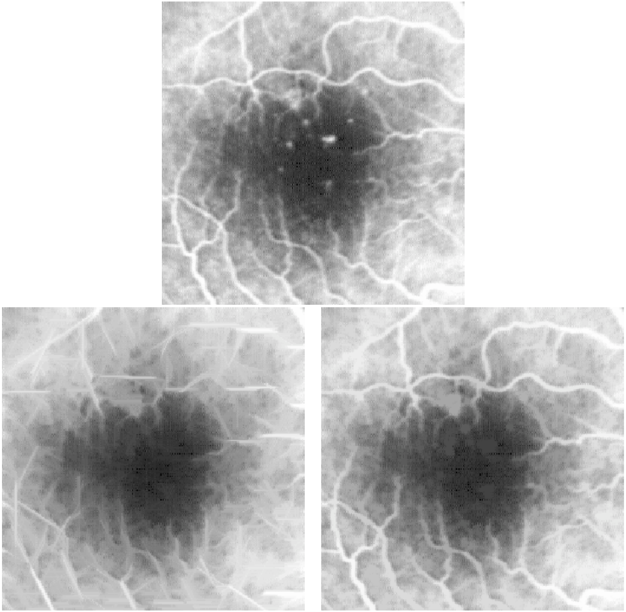


Numerical reconstruction: example



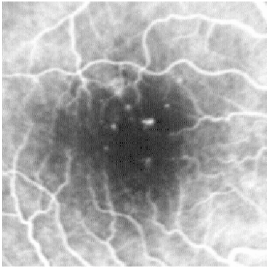
Opening by reconstruction: examples



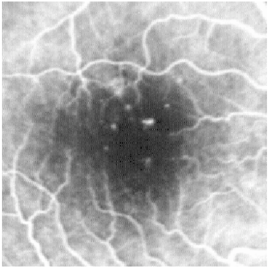


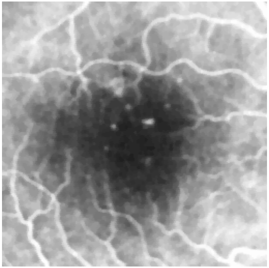
Union of openings by segments of length 20 and reconstruction

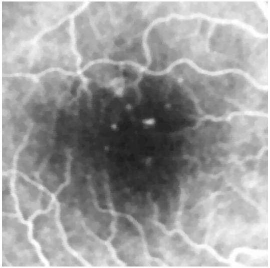
Application to alternate sequential filters



Application to alternate sequential filters

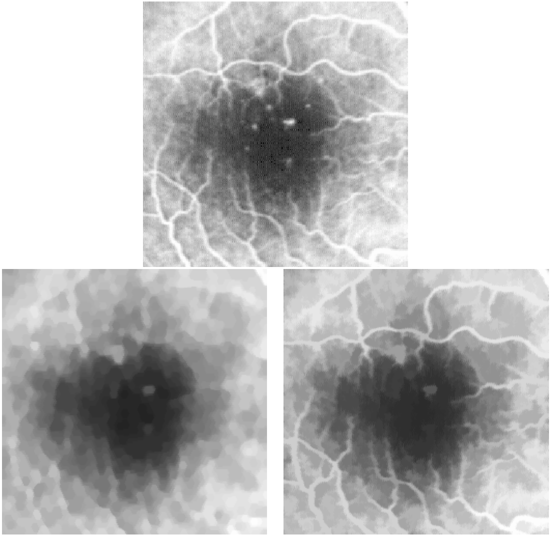






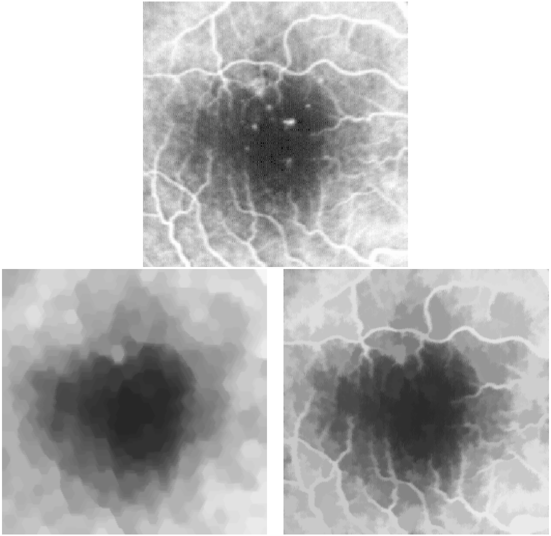
ASF with an hexagon (maximal size = 1) - Right: with reconstruction

Application to alternate sequential filters



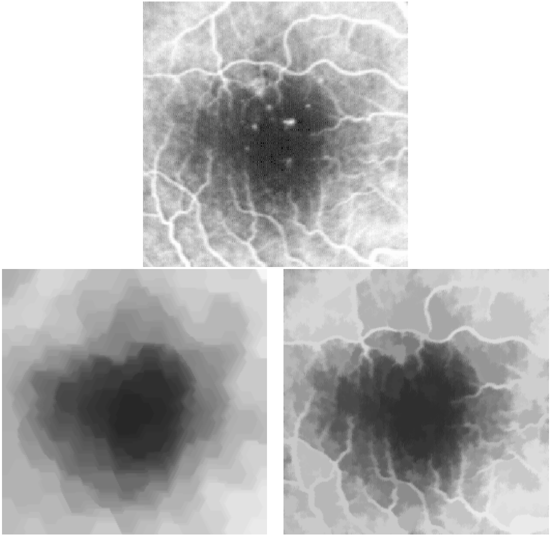
ASF with an hexagon (maximal size = 3)

Application to alternate sequential filters



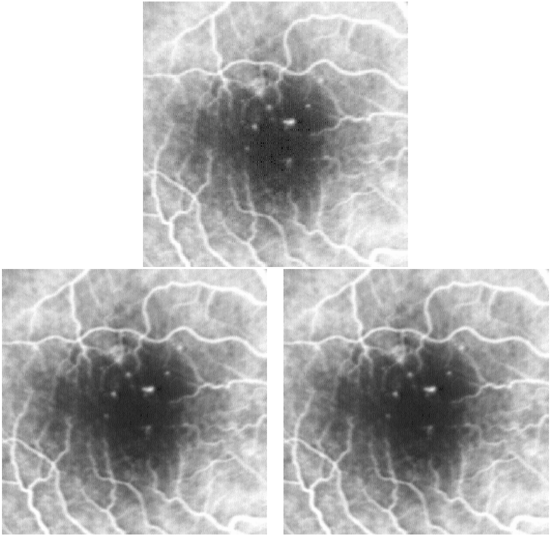
ASF with an hexagon (maximal size = 5)

Application to alternate sequential filters



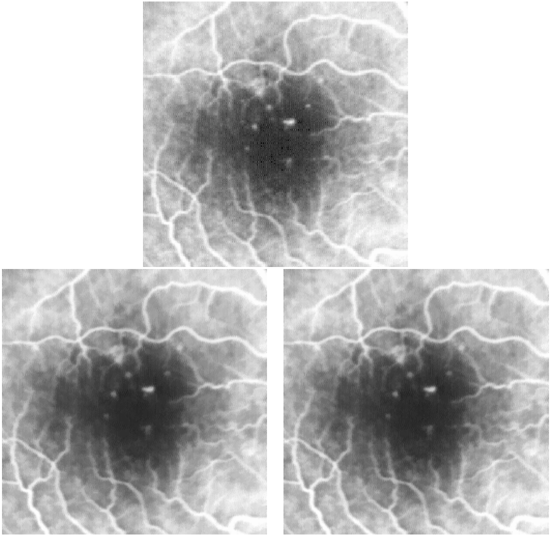
ASF with an hexagon (maximal size = 9)

Application to alternate sequential filters



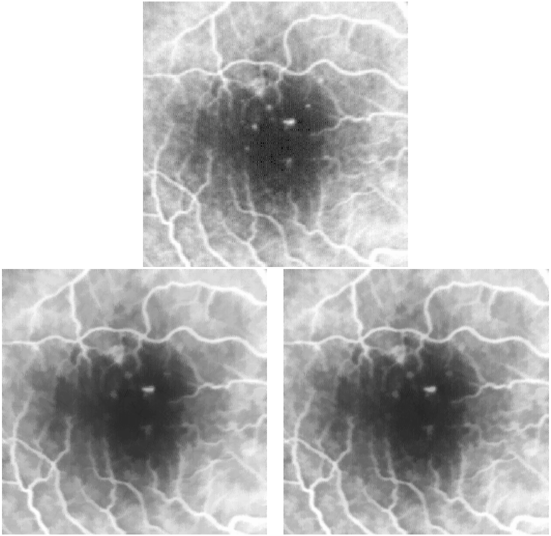
ASF with segments (maximal size = 1)

Application to alternate sequential filters

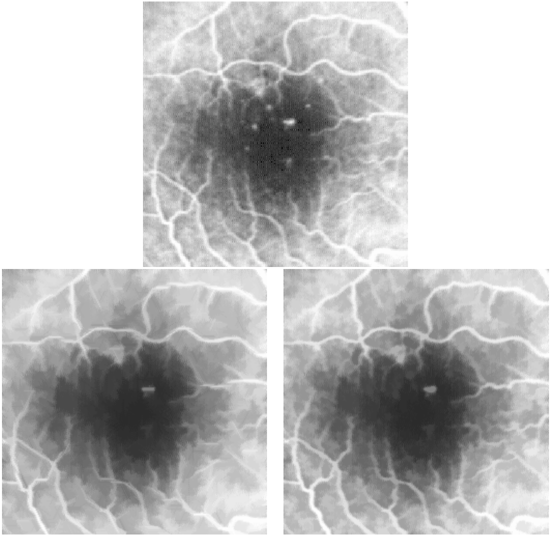


ASF with segments (maximal size = 3)

Application to alternate sequential filters



ASF with segments (maximal size $= 5$)



ASF with segments (maximal size $= 9$)

Regional maxima

X regional maximum of f if

$$\forall x \in X, f(x) = \lambda \text{ et } X = CC(f_\lambda)$$

Computation of regional maxima:

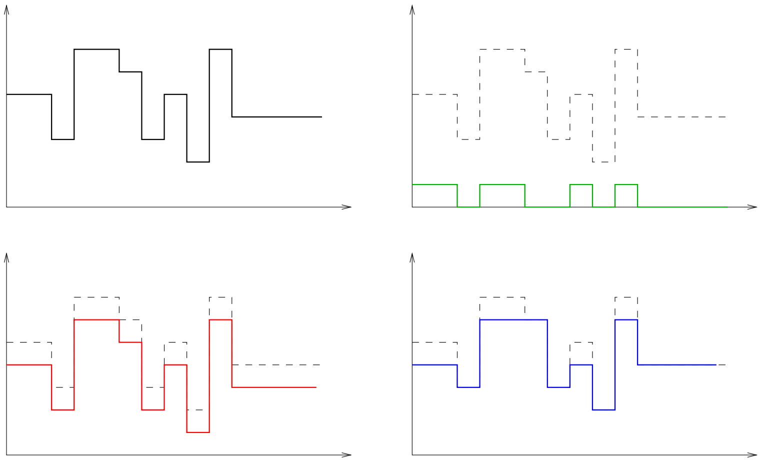
$$f - D_f^\infty(f - 1)$$

h-maxima (gray level dynamics): regional maxima of

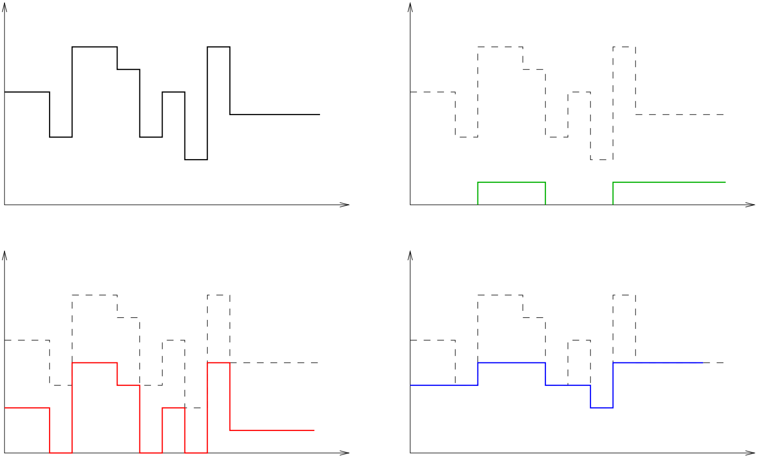
$$D_f^\infty(f - h)$$

$\Rightarrow$ robust maxima

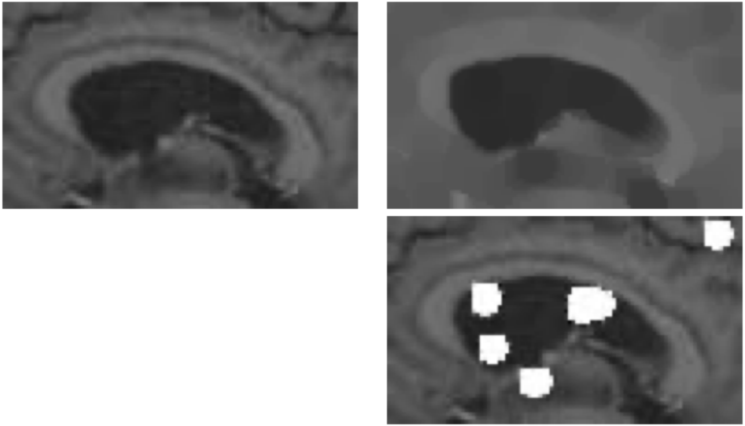
Regional maxima: example



Robust maxima: example



Regional minima: example



Skeleton by influence zones

$X = \bigcup_i X_i$

Influence zone of X_i in X^C :

$$ZI(X_i) = \{x \in X^C \mid d(x, X_i) < d(x, X \setminus X_i)\}$$

Skeleton by influence zones:

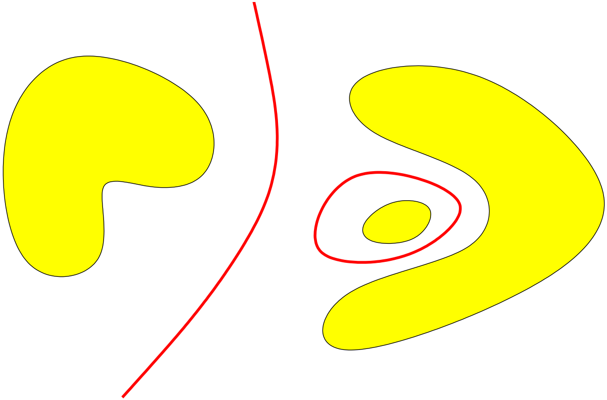
$$\text{Skiz}(X) = (\bigcup_i ZI(X_i))^C$$

= generalized Voronoï diagram

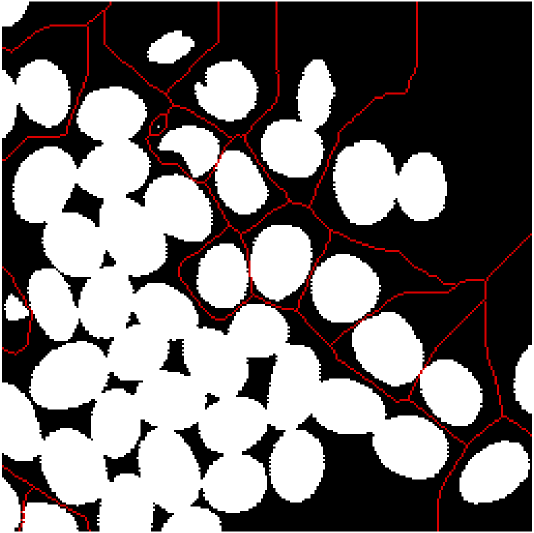
Properties:

- $\text{Skiz}(X) \subseteq Skel(X^C)$
- Skiz is not necessarily connected (even if X^C is)

Skeleton by influence zones: examples



Skeleton by influence zones: examples



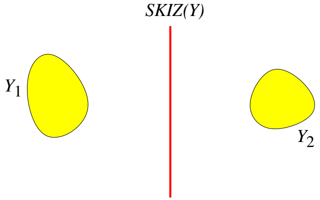
$Y = \cup_i Y_i$

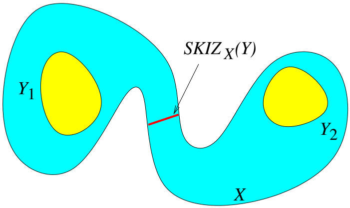
Geodesic influence zone of Y_i conditionally to X :

$$ZI_X(Y_i) = \{x \in X, d_X(x, Y_i) < d_X(x, Y \setminus Y_i)\}$$

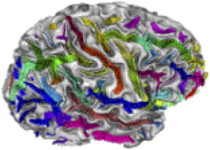
Geodesic skeleton by influence zones:

$$SKIZ_X(Y) = X \setminus \bigcup_i ZI_X(Y_i)$$



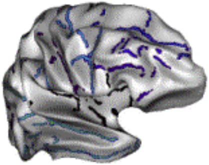


Cortex segmentation (PhD of Arnaud Cachia)

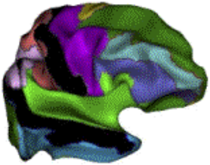


Segmentation et reconnaissance automatique des **sillons**

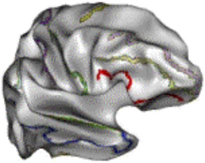
Rivière00



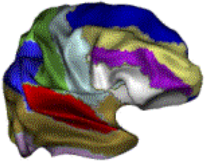
Définition sur la surface corticale des **sillons-frontières**



Calcul des **zones d'influences sulcales**

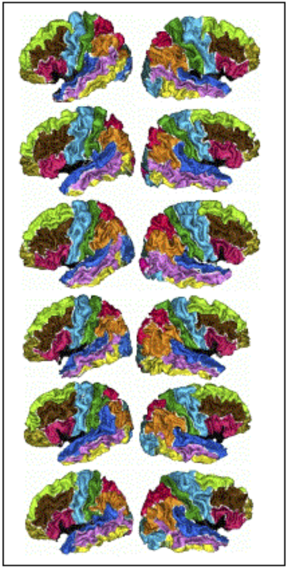


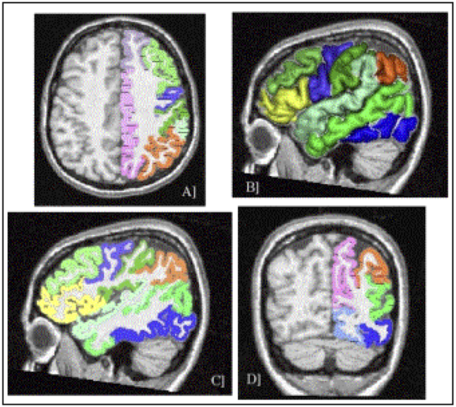
Définition des **graines gyrales** (extraction et sélection des frontières)



Parcellisation en **gyri** (2D et 3D)

Cortex segmentation (PhD of Arnaud Cachia)



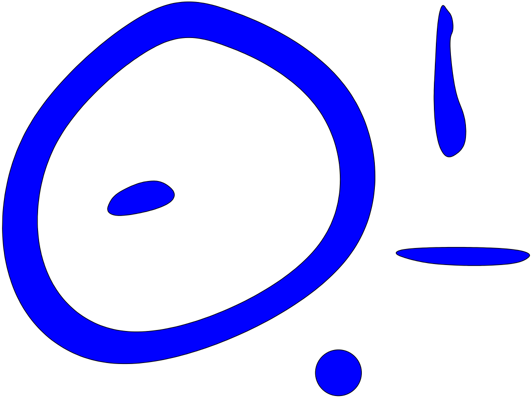


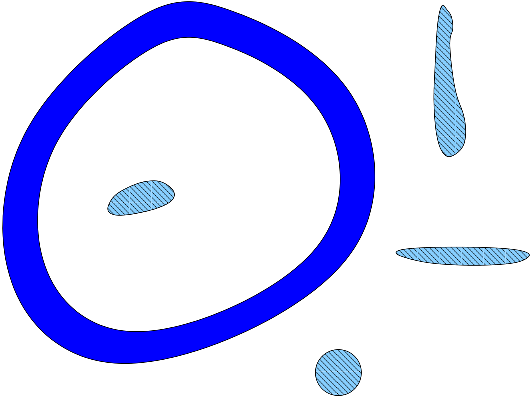
Parcellisation volumique
(diagramme de Voronoï calculé
dans le ruban cortical 3D)

Connected filters

- Objective: make the image simpler
- Morphological filter that
  - preserves contours
  - is independent of the contrast
  - acts on connected components
- First example: surfacic opening

$$\gamma_\lambda(f) = \bigvee_i \{\gamma_{B_i}(f) \mid B_i \text{ connected and } S(B_i) = \lambda\}$$





Connected filters on gray level images

- For increasing operations
- Cut by cut computation

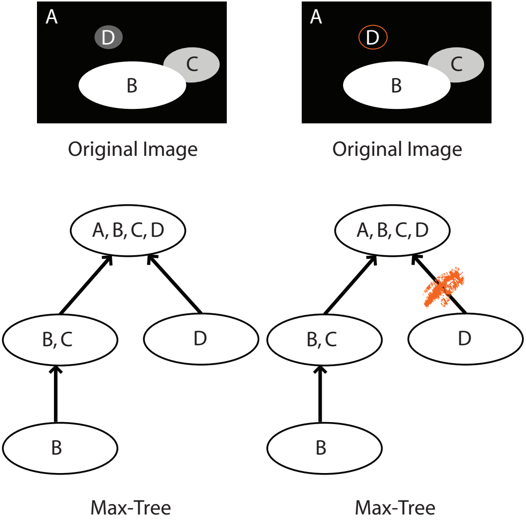
$$T_h(f) = \{x \mid f(x) \geq h\}$$

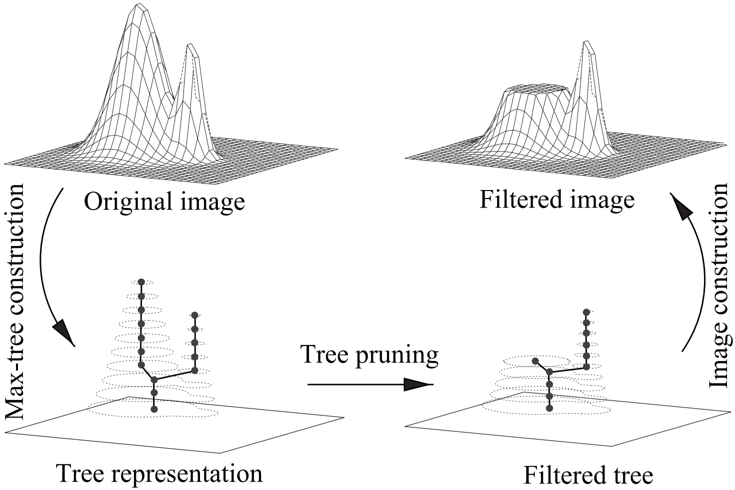
$$(\gamma^A(f))(x) = \sup\{h \mid x \in \Gamma^A(T_h(f))\}$$

Surfacic opening   Surfacic closing

- More complex for non increasing operators

Max-tree (or min-tree) representation





from (Salembier ITIP 00)

Original Image

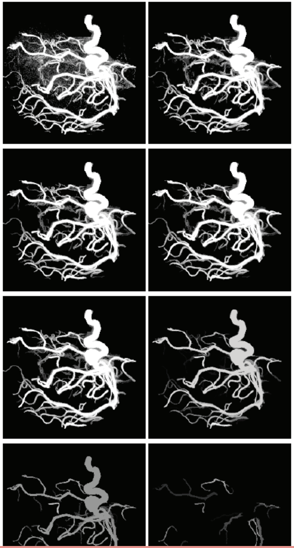
Area Opening

Moment of Inertia Opening

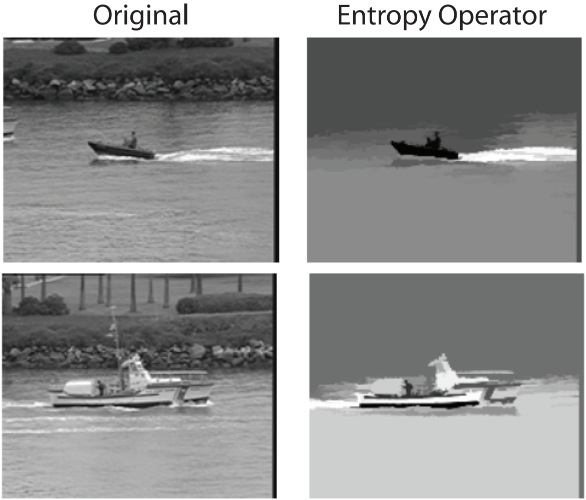
from (Wilkinson ISMM 00)

Examples

Filtering depending on elongation (Meijster, 2002) :



Entropy criterion (Salembier, 1998) :



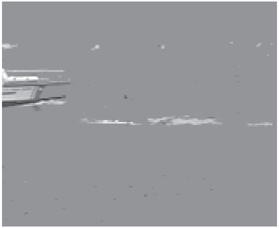
from (Salembier ITIP 98)

Movement analysis (Salembier, 1998) :

Original frame

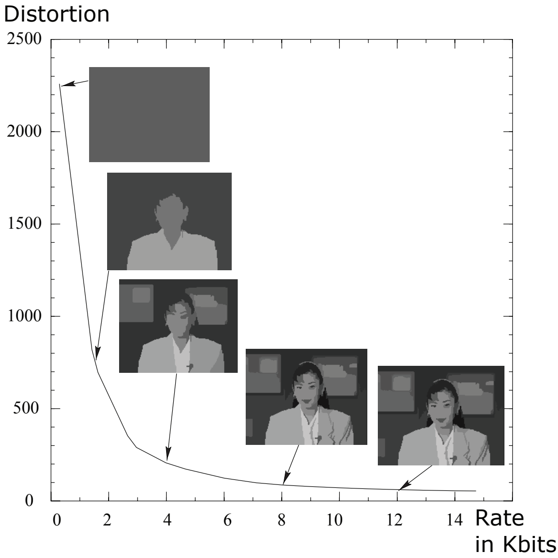
Objects with translation (0,0)

Objects with translation (2,0)



Remaining Objects

Compression (Salembier, 2000) :

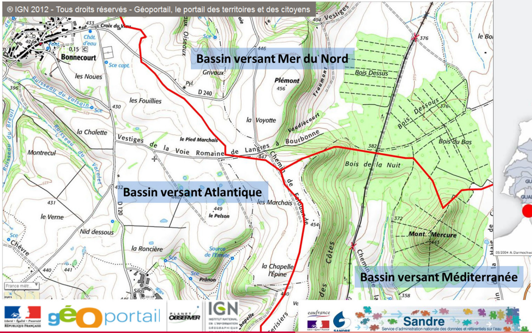


from (Salembier ITIP 00)

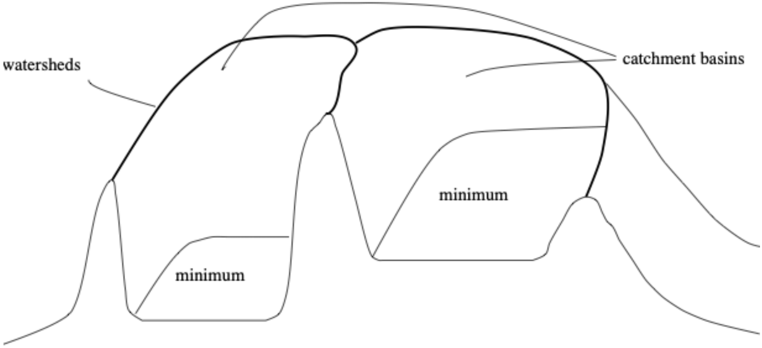
Segmentation based on mathematical morphology

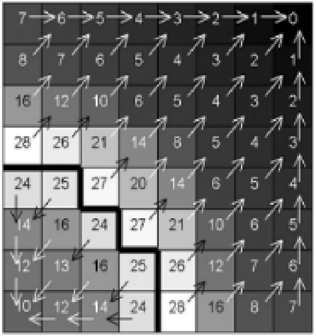
- Criteria for image segmentation:
  - simplicity
  - regularity
  - fidelity to the data
- Two morphological paradigms:
  - flat zones (using connected filters)
  - catchment basins and watersheds


Ligne de partage
des eaux


® IGN 2012 - Tous droits réservés - Géoportail, le portail des territoires et des citoyens
Bassin versant Mer du Nord
Bassin versant Atlantique
Bassin versant Méditerranée
Bonnecourt
Plémont
les Noues
les Fouillies
la Voyotte
la Chalette
Montrecul
le Verne
Nid dessous
la Roncière
Bois Dessus
Bois du Bas
Bois de la Nuit
Mont Mercure
Vestiges de la Voie Romaine de Langres à Bourbonne
géoportail
IGN
Sandre


MER DE BARENTS
MER BALTIQUE
MER DU NORD
OCÉAN ATLANTIQUE
MÉDITERRANÉE
ADRIATIQUE
MER NOIRE
MER CASPIENNE
DVINA SEPTENTRIONALE
VOLGA
DON
DNIEPR
DVINA OCCIDENTALE
NIÉMEN
NARVA
NEVA
VISTULE
ODER
ELBE
WESER
RHIN
MEUSE
SEINE
LOIRE
GARONNE
RHÔNE
PÔ
DANUBE
BUG
DNIESTR
DOURO
ÈBRE
TAGE
GUADIANA
GUADALQUIVIR
TIBRE
TAMISE
TRENT
SEVERN
OUSE
Bassins hydrographiques et principales lignes de partage des eaux européennes
Fleuve
Bassin hydrographique
Principales lignes de partage des eaux européennes


watersheds
catchment basins
minimum
minimum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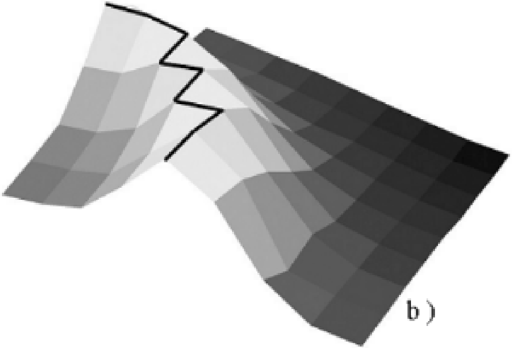
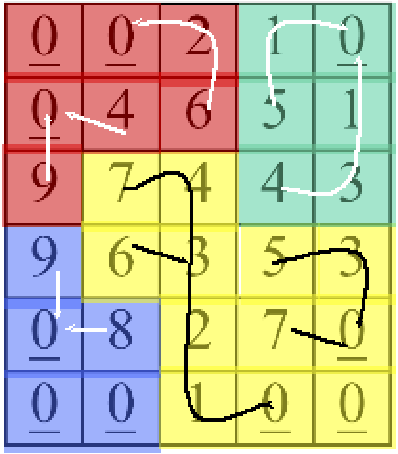
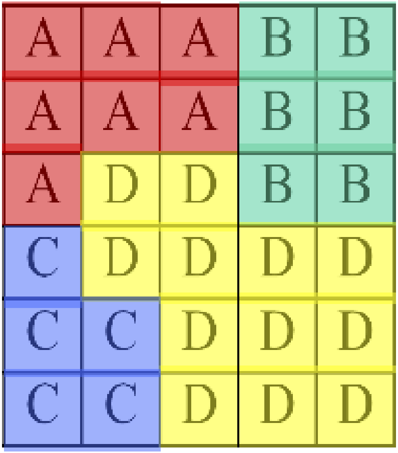
8 7 6 5 4 3 2 1
16 12 10 6 5 4 3 2
28 26 21 14 8 5 4 3
24 25 27 20 14 6 5 4
14 16 24 27 21 10 6 5
12 13 16 25 26 12 7 6
10 12 14 24 28 16 8 7

a)



b)





Several solutions...

Watersheds: definition

Steepest descent:

$$Desc(x) = \max\{\frac{f(x) - f(y)}{d(x,y)},\ y \in V(x)\}$$

Ramp of a path $\pi = (x_0, ... x_n)$:

$$T_f(\pi) = \sum_{i=1}^{n} d(x_{i-1}, x_i) Cost(x_{i-1}, x_i)$$

with

$$Cost(x,y) = \begin{cases} Desc(x) & \text{if } f(x) > f(y) \\ Desc(y) & \text{if } f(y) > f(x) \\ (Desc(x) + Desc(y))/2 & \text{if } f(y) = f(x) \end{cases}$$

Topographic distance

$$T_f(x, y) = \inf\{T_f(\pi), \pi = (x_0 = x, x_1, ..., x_n = y)\}$$

(equals 0 on a plateau)

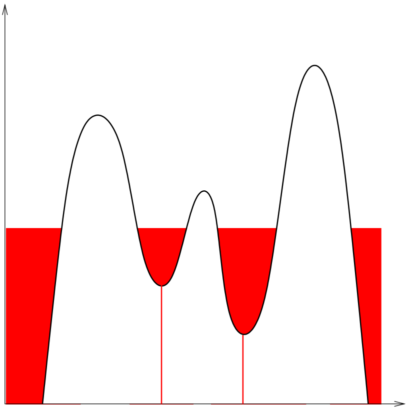
Catchment basin associated with the regional minimum M_i:

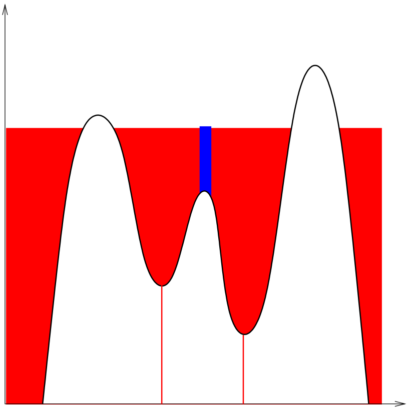
$$CB(M_i) = \{x \mid \forall j \neq i, T_f(x, M_i) + f(M_i) < T_f(x, M_j) + f(M_j)\}$$

Watersheds:

$$WS(f) = [\cup_i CB(M_i)]^C$$

Approach by immersion





Construction of the watersheds

f such that $f(x) \in [h_{\min}, h_{\max}]$, $f^h = \{x,\ f(x) \leq h\}$

$$\begin{aligned} X_{h_{\min}} &= f^{h_{\min}} \\ X_{h+1} &= MinReg_{h+1}(f) \cup ZI_{f^{h+1}}(X_h) \\ CB &= X_{h_{\max}} \\ WS(f) &= X^C_{h_{\max}} \end{aligned}$$

Illustration of the algorithm

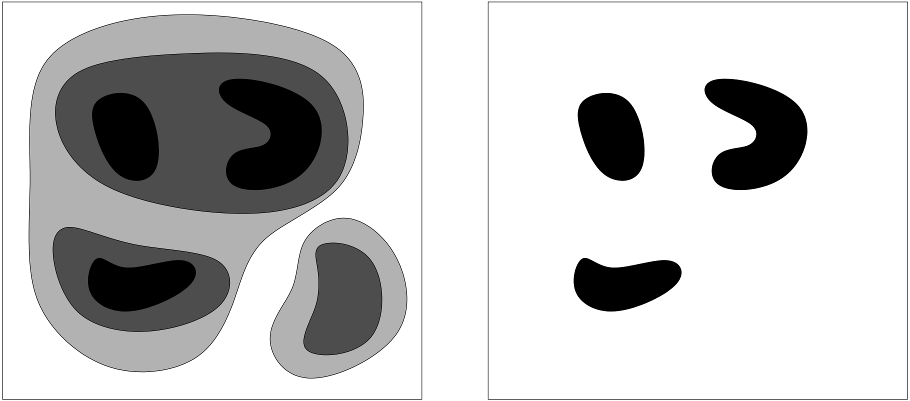
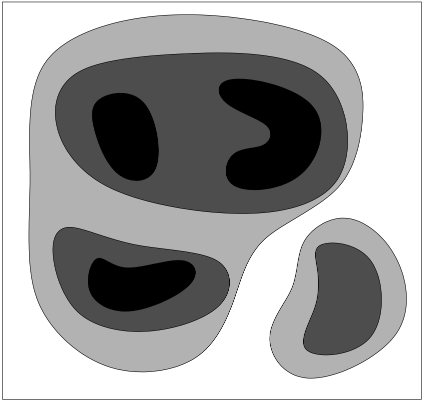


Illustration of the algorithm

Illustration of the algorithm



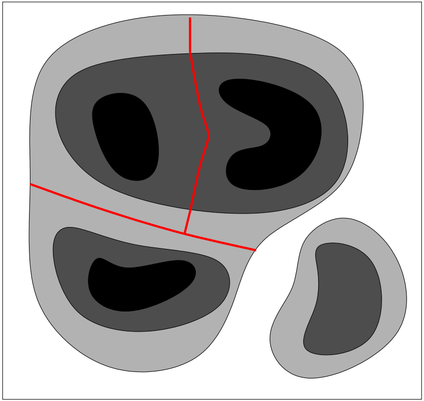
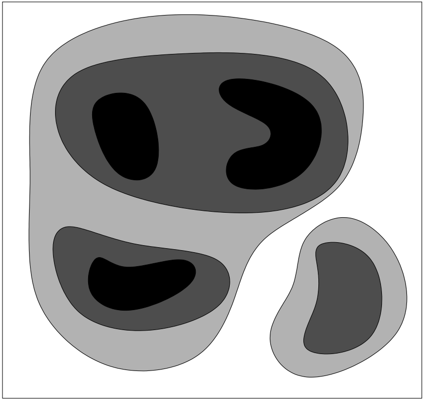
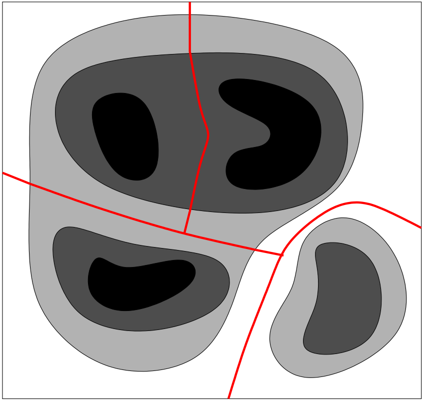
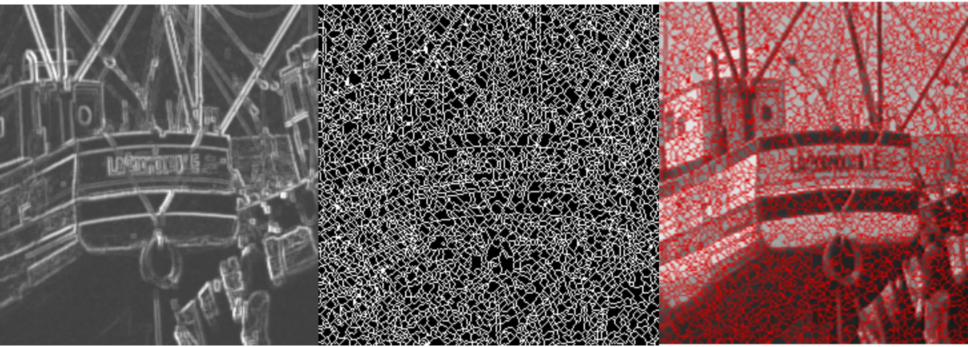


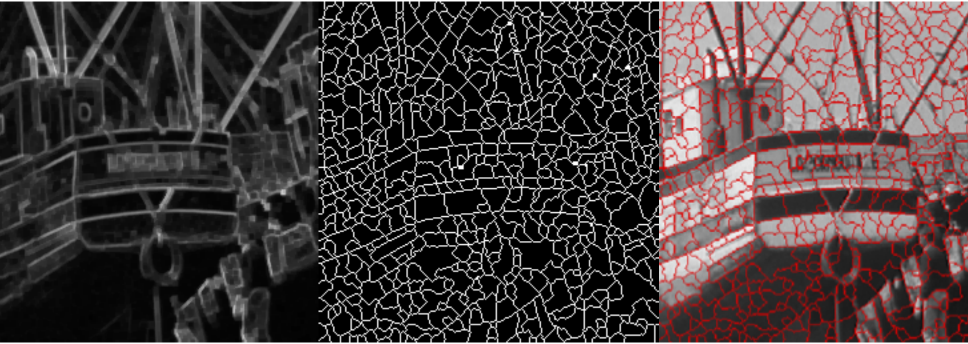
Illustration of the algorithm



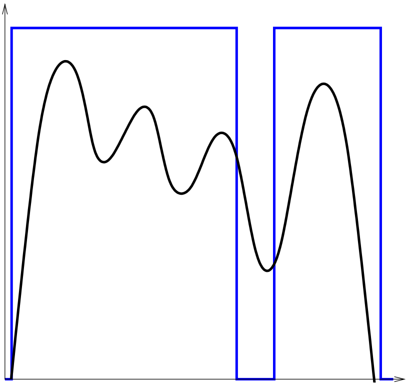


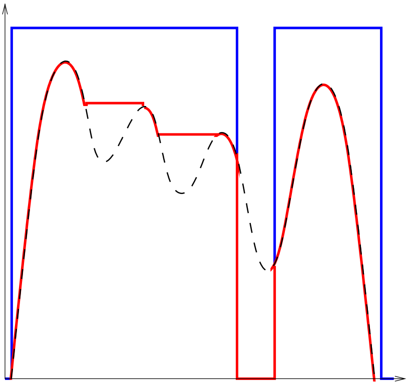
Watersheds and oversegmentation





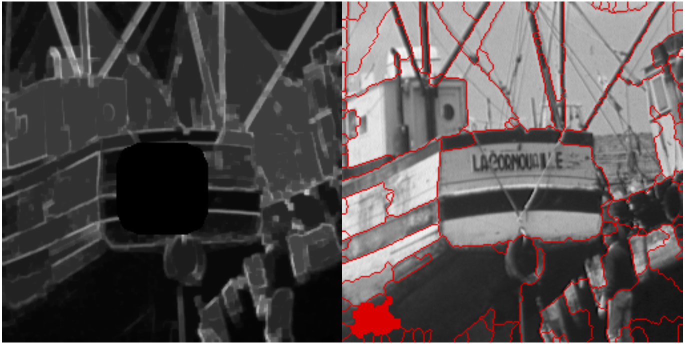
Geodesic erosion in order to impose markers

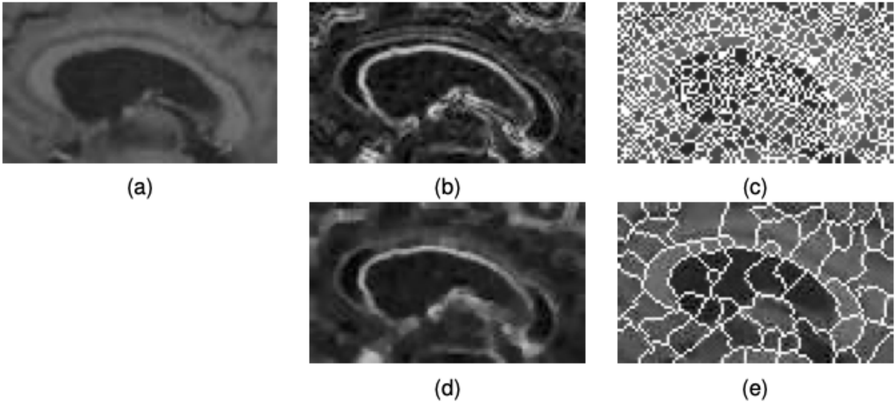




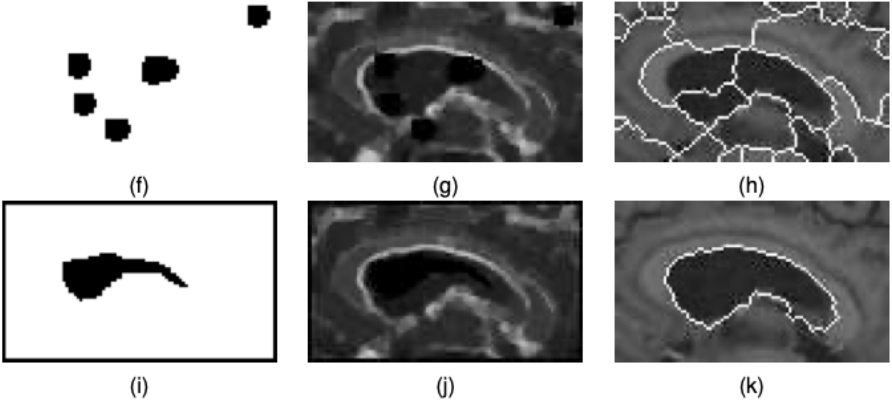
Watersheds constraint by markers

f: function on which watersheds should be applied
g: marker function (selects regional minima)
Reconstruction: $E_{f \wedge g}(g, B_\infty)$ (only the selected minima)





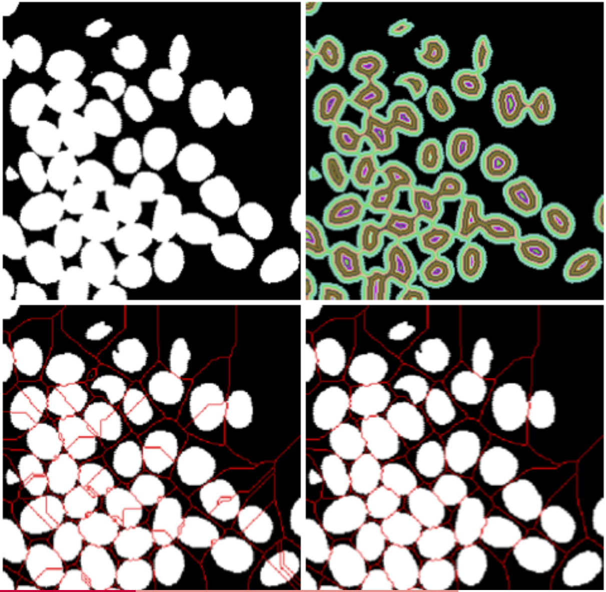
(a) Original image (from a brain MRI). (b) Morphological gradient. (c) Watersheds. (d) Closing (size 1) of the gradient. (e) Watersheds applied on the closing of the gradient.



(f) Markers from regional minima. (g) Reconstruction of the gradient. (h) Watersheds. (i) Markers *inside* the ventricles and on the image border. (j) Reconstructed gradient. (k) Watersheds providing the right contours of the ventricles.

Interactive marker-based segmentation

Separation of connected binary objects



Watershed as an energy minimization problem

Minimum of the energy defined as (Boomgard, 2000):

$$E = \sum_i \int \int_{D_i} (f(D_i) + T_f(x, D_i))dx$$

D_i = regional minimum
$f(D_i)$ = value of the regional minimum

Regularization of watersheds

- using closing
- watersnakes (Boomgard, 2003): additional term controlling the length of the contours

$$E = \sum_i \left(\int \int_{D_i} (f(D_i) + T_f(x, D_i))dx + \beta \int_{\partial D_i} ds \right)$$

- geometrical constraints
- viscous flooding (Vachier and Meyer)

Watersnakes: examples (Boomgard, 2003)

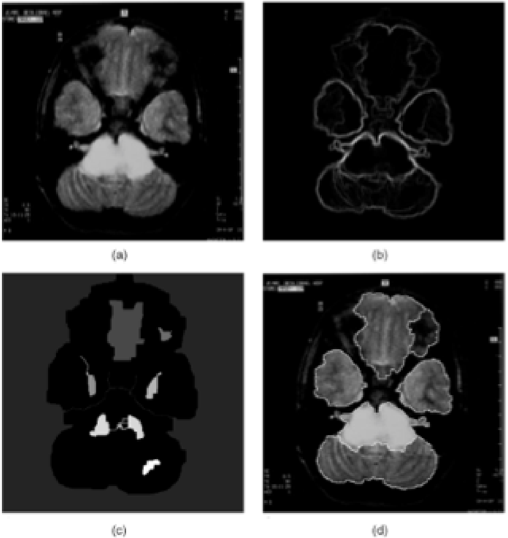


Fig. 6. (a) A brain image. (b) The relief computed from morphological gradient. (c) The markers extracted. (d) The result of the original watershed segmentation, shown for comparison.

Watersnakes: examples (Boomgard, 2003)

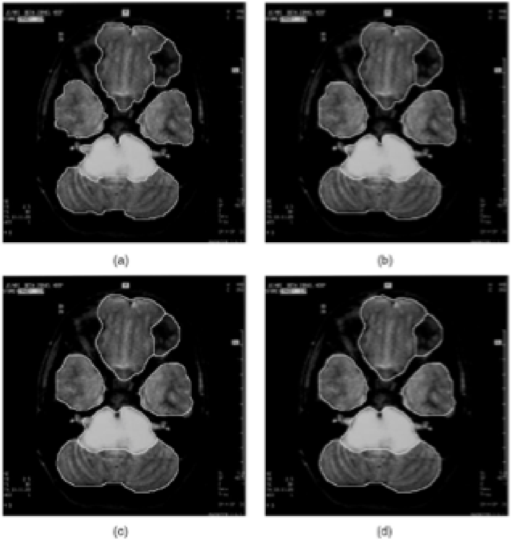


Fig. 7. The segmentation result of the watersnake algorithm based on energy discretization with (a) $\beta = 10$, (b) $\beta = 50$, (c) $\beta = 100$, and (d) $\beta = 150$. Note, in comparison with the original watershed segmentation in Fig. 6d, that the results in figure are smoother, but still identify the main objects.

Watersnakes: examples (Boomgard, 2003)

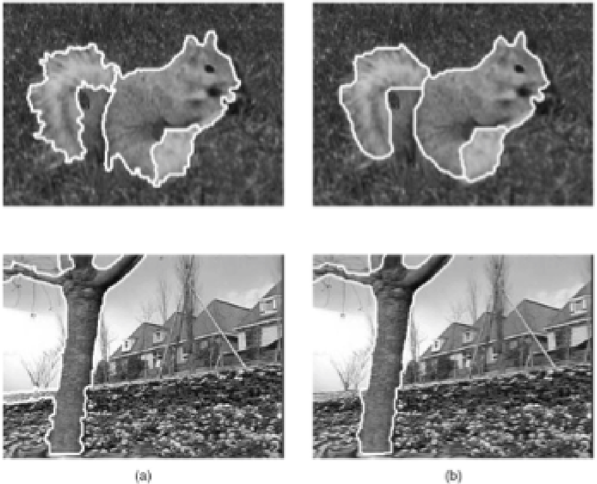
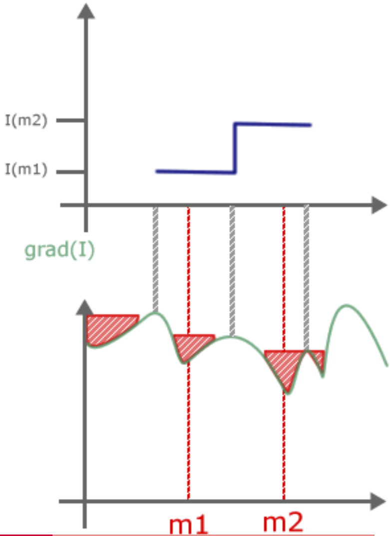


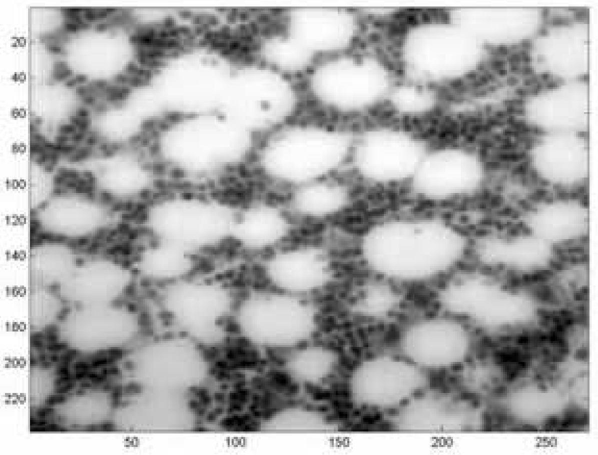
Fig. 8. Segmentation by (a) watershed and (b) watersnake ($\beta = 50$). In the bottom row, the result is shown for the object of interest only.

Hierarchical watersheds

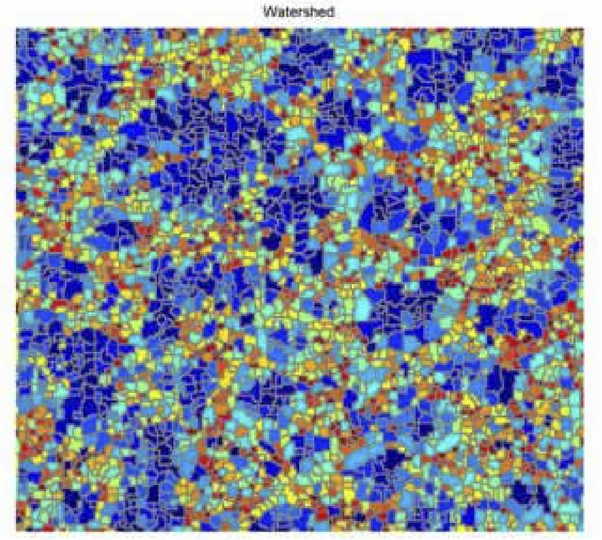
Iterative filling of catchment basins:



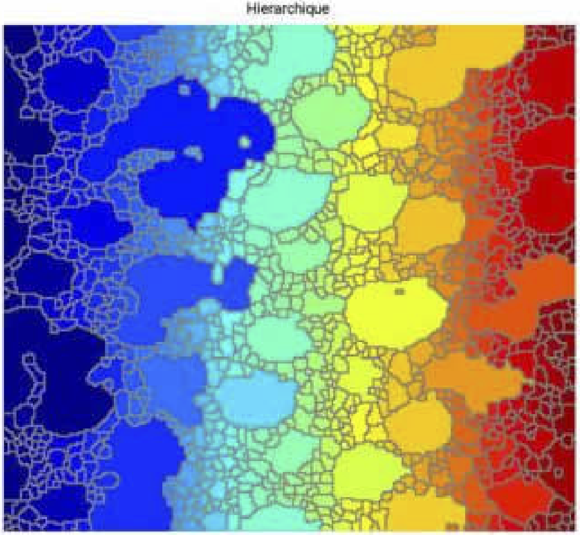
Hierarchical watersheds



Hierarchical watersheds



Hierarchical watersheds



Hit-or-Miss Transformation (HMT)

Structuring element: $T = (T_1, T_2)$, with $T_1 \cap T_2 = \emptyset$

HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

Thinning (if $O \in T_1$):

$$X \circ T = X \setminus X \otimes T$$

Thickening (if $O \in T_2$):

$$X \odot T = X \cup X \otimes T$$

For $T' = (T_2, T_1)$:

$$X \circ T = (X^C \odot T')^C$$

Applications of HMT

- end points
- multiple points
- convex hull
- homotopic skeleton
- ...

Outlook

- And many other operations!
- Current trend: mathematical morphology associated with artificial neural networks
  - additional input
  - post-processing
  - non-linear convolutions
  - inside the cost (loss) function to be optimized
  - ...