Digital Representations

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Content

- Tessellations
- Digital topology
- Representation of geometrical entities
- Distance function

Digital representation of images

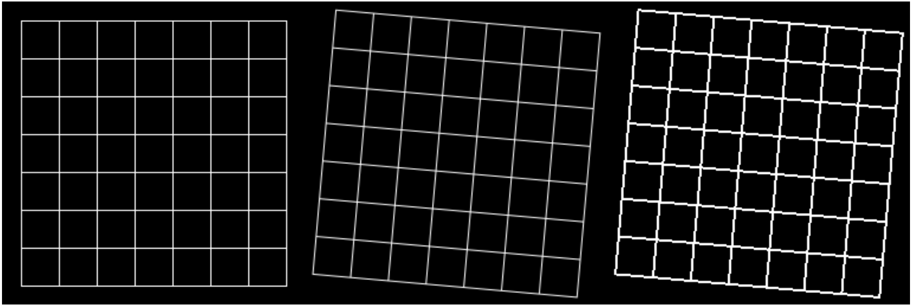
Digital image:

- representation as an array
- sampling from $\mathbb{R}^2$ or $\mathbb{R}^3$ into $\mathbb{Z}^2$ or $\mathbb{Z}^3$

Two possible approaches to process points in $\mathbb{Z}^n$:

- embed $\mathbb{Z}^n$ in $\mathbb{R}^n$, then apply operations and transformations in the continuous space
- definition of operations and transformations directly in the digital space
  - definitions?
  - preservation of the expected effects?
  - preservation of the properties?

A simple example: rotation

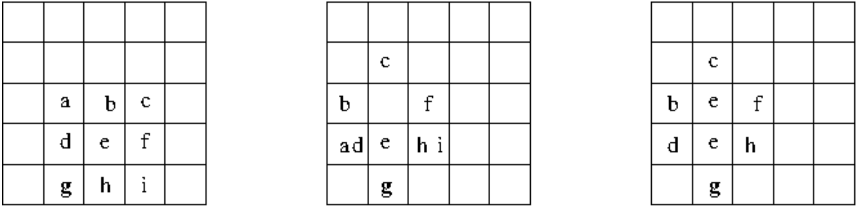


$$(x', y') = R(x, y)$$

Issues:

- (x, y) = integer coordinates $\Rightarrow (x', y')$?
- Computation?
- Properties?

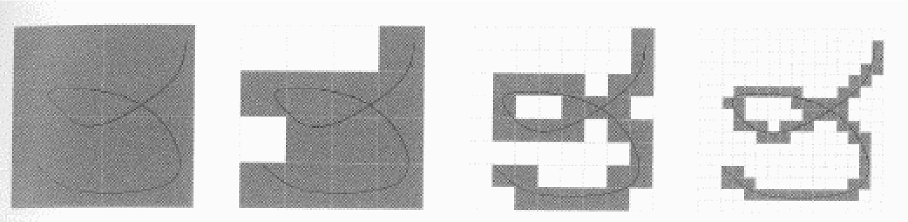
$\pi/4$ rotation: $x' = (x - y)\frac{\sqrt{2}}{2} \quad y' = (x + y)\frac{\sqrt{2}}{2}$



Direct transformation

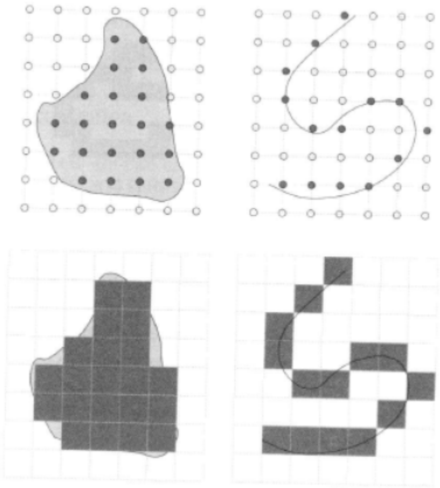
Inverse transformation (closest point interpolation)

Topology and resolution



Source: D. Cœurjolly, A. Montanvert, J. M. Chassery (2007)

Curve digitization



Source : D. Cœurjolly, A. Montanvert, J. M. Chassery (2007)

Tessellations

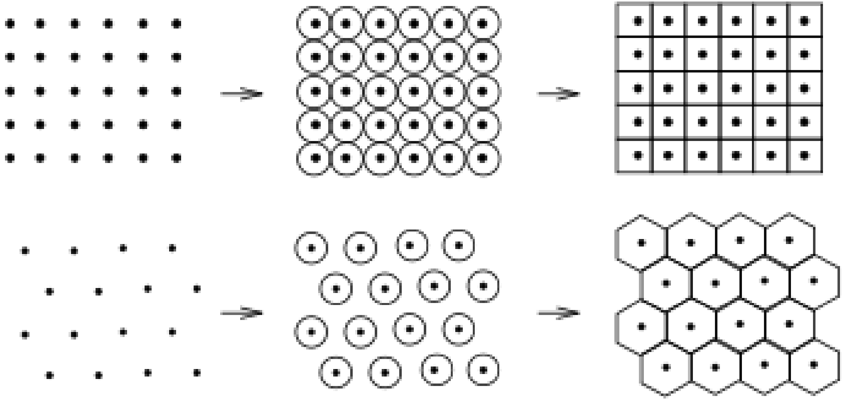
Tessellation = partition of the continuous space $\mathbb{R}^n$ into elementary cells

Constraints:

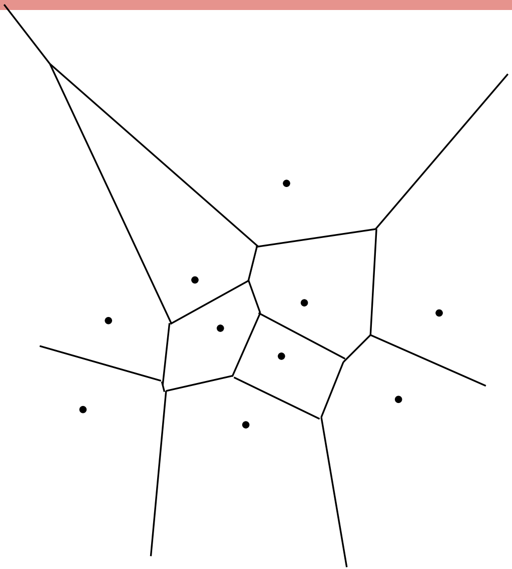
- physical sensors (regularity)
- usage of the representation (regularity, simplicity)

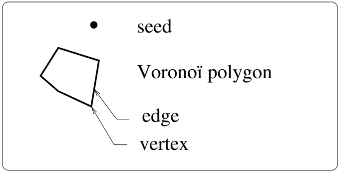
1. Tessellation from point distribution
   - distribution of points P
     - regular $\Rightarrow$ classical grids
     - irregular $\Rightarrow$ Voronoï diagram
   - attribution of a cell V_P to each point
2. Tessellation from cell juxtaposition
   - prior definition of a cell model V_P
   - juxtaposition of V_P so as to build a partition
   - constraints:
     - V_P convex and regular
     - vertices in contact with other vertices only

Regular distributions

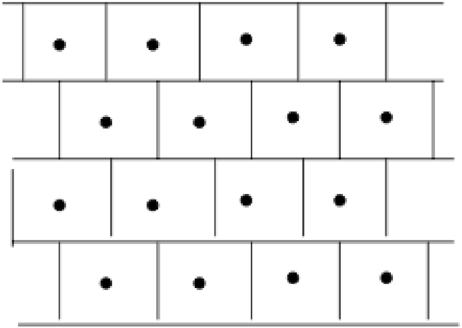


Irregular distributions: Voronoï diagram

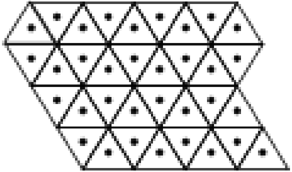




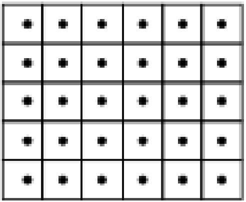
An excluded configuration



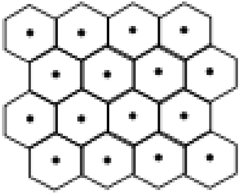
Regular tessellations of the plane



triangular



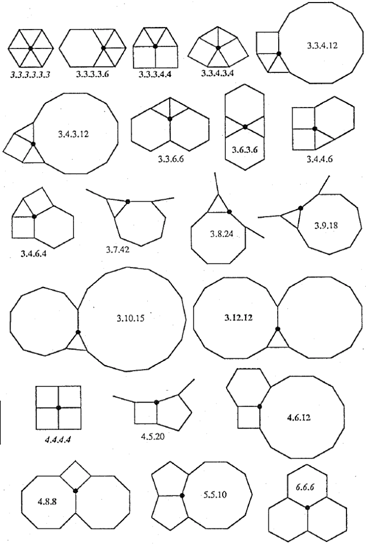
square



hexagonal

Semi-regular tessellations

regular

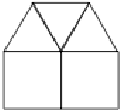


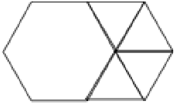
regular

regular

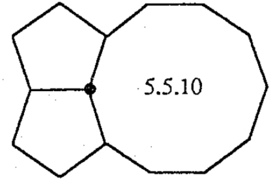
Examples

Admissible:

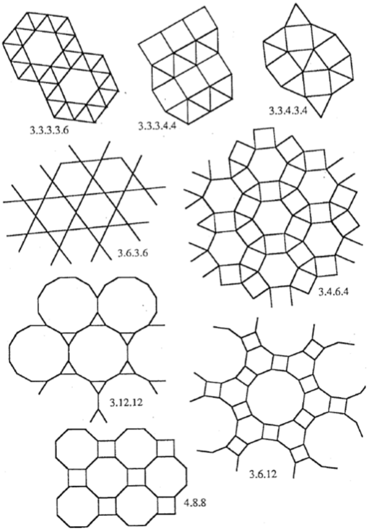




Non-admissible:

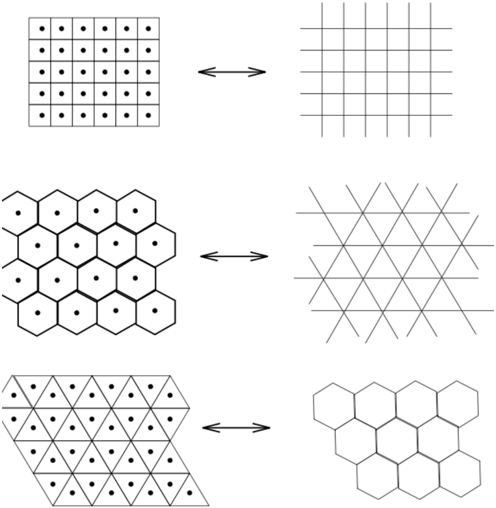


Admissible semi-regular tessellations



A complex tessellations (Escher)...

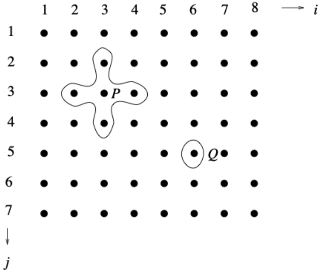
Duality between tessellations and mesh (or grid)

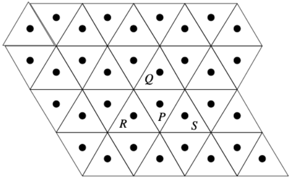


Digital topology

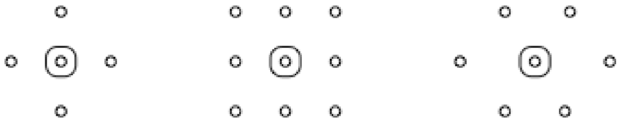
- Classical topology in a countable set of points:
  - every point is an open set of the topology
  - not well adapted to the representation of connected sets
- Direct definition of a topological basis
  - possible on triangular and square tessellations, not on hexagonal tessellations
  - depends on point localization
  - does not satisfy Jordan theorem
- Direct definition of elementary neighborhood
  - digital connectivity
  - image = graph

Direct definition of a topological basis: examples

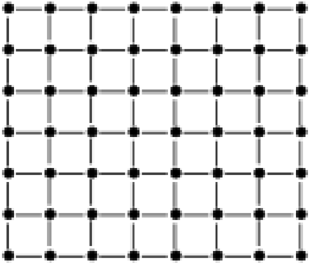


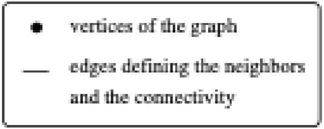


Elementary neighborhood

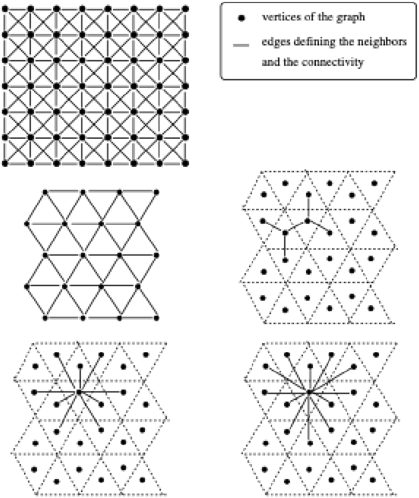


4-connectivity graph

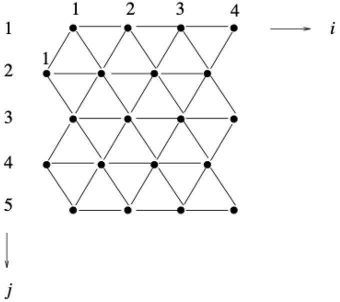




Different grids and associated connectivities

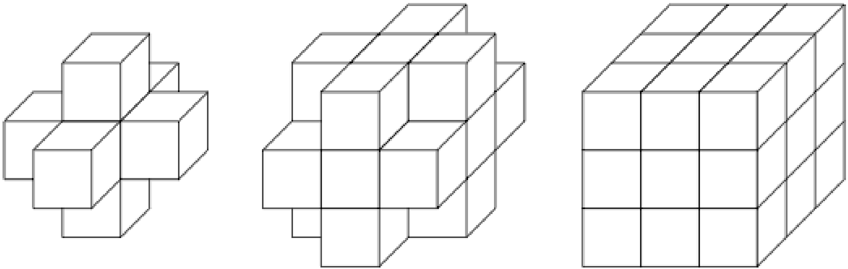


Neighbor coordinates on a hexagonal grid



- for j even: $(i-1, j-1)$, $(i, j-1)$, $(i-1, j)$, $(i+1, j)$, $(i-1, j+1)$, $(i, j+1)$,
- for j odd: $(i, j-1)$, $(i+1, j-1)$, $(i-1, j)$, $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$.

Elementary neighborhood on a 3D cubic grid



Path and connected component

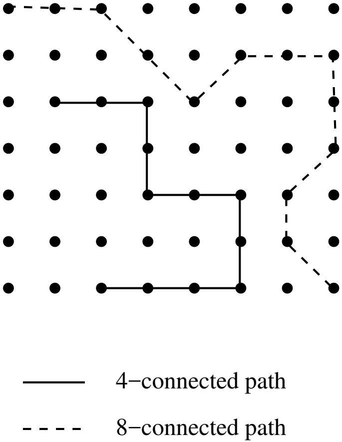
- 4-connected path = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

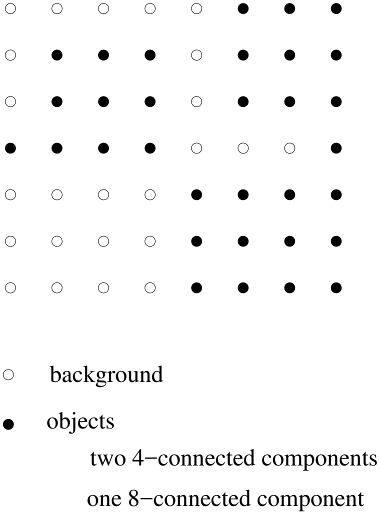
$$\forall k, 1 \leq k < n, \ |i_k - i_{k+1}| + |j_k - j_{k+1}| \leq 1$$

- 8-connected path = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

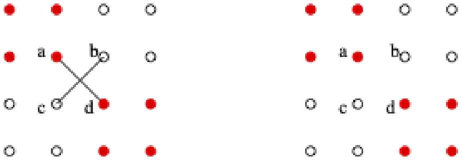
$$\forall k, 1 \leq k < n, \ \max(|i_k - i_{k+1}|, |j_k - j_{k+1}|) \leq 1$$

- 4-connected component = set of points $\mathcal{S}$ such that for any (P, Q) in $\mathcal{S}$, there exists a 4-connected path from P to Q, included in $\mathcal{S}$, and maximal for this property.
- 8-connected component = set of points $\mathcal{S}$ such that for any (P, Q) in $\mathcal{S}$, there exists a 8-connected path from P to Q, included in $\mathcal{S}$, and maximal for this property.


4−connected path
8−connected path

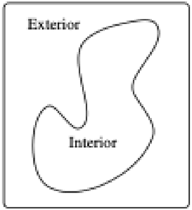

background
objects
two 4−connected components
one 8−connected component

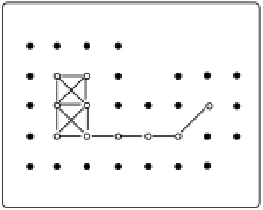
Topological paradox



Jordan theorem

- Continuous case: any simple and closed curve divides the space into two connected components, one inside the curve and one outside.
- Digital case: duality between 4-connectivity and 8-connectivity on a square grid
  - 4-connected curve ⇔ 8-connected background,
  - 8-connected curve ⇔ 4-connected background.
- Digital case on a hexagonal grid: 6-connectivity for both objects and background (no topological problem).
- Extension to 3D.





Some definitions in the digital case

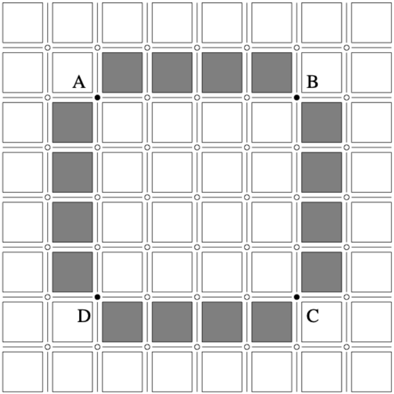
- Simple and closed 4-connected path: 4-connected path $(A_0, ..., A_n)$ such that $n \geq 4$, $A_i = A_j$ iff $i = j$, and A_i 4-neighbor of A_j iff $i = j \pm 1[n+1]$
- Horizontal half-line from $M = (a, b)$:

$$H_M = \{(a+k, b), k = 0, 1, 2...\}$$

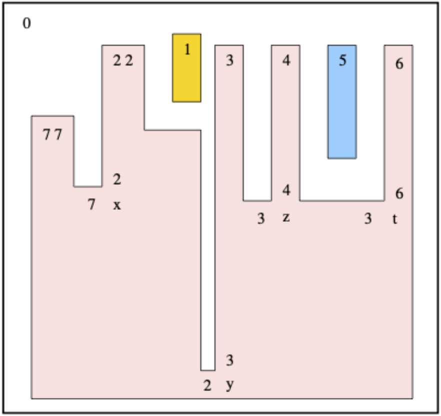
- Inside A: set of points M such that H_M crosses A an odd number of times.
- Outside A: set of points M such that H_M crosses A an even number of times.

$\Rightarrow$ proof of the digital version of Jordan theorem.

Cellular complexes



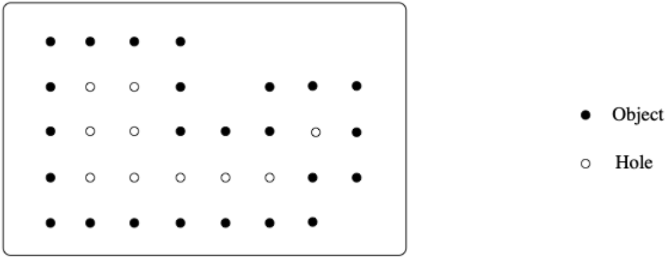
Connected component labeling



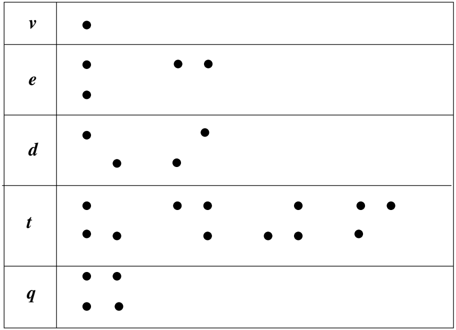
initial label	pointer	final label
0		0
1		1
2		2
3 →	2	2
4 →	3	2
5		3
6 →	3	2
7 →	2	2

Example of topological characteristic: Euler number

- Number of connected components N_{cc}
- Number of holes N_t
- Euler number $E = N_{cc} - N_t$



- 8-connected objects and 4-connected holes: $N_{cc} = 1$ and $N_t = 2$, hence $E = -1$
- 4-connected objects and 8-connected holes: $N_{cc} = 1$ and $N_t = 1$, hence $E = 0$



- 8-connected objects and 4-connected holes:

$$E = v - e - d + t - q$$

- 4-connected objects and 8-connected holes:

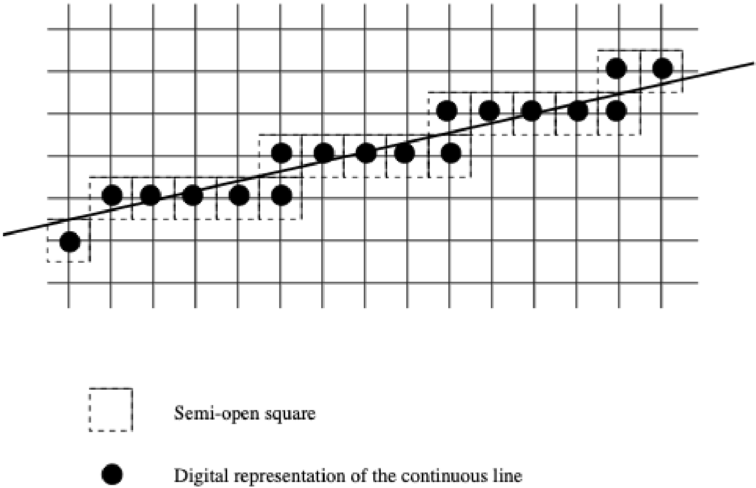
$$E = v - e + q$$

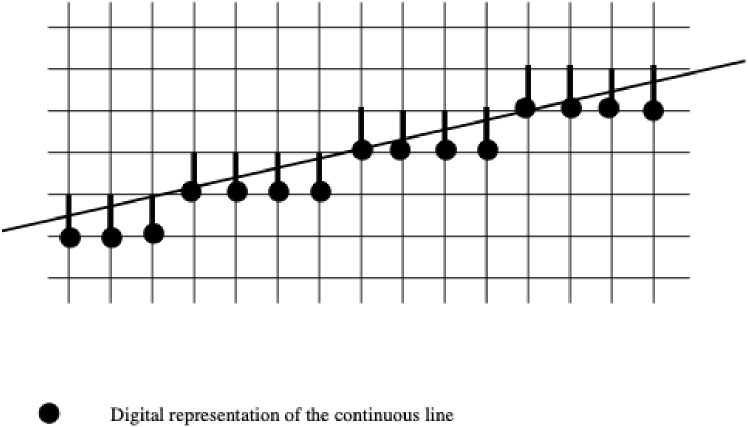
Digital geometry

- How to go from the continuous domain to the digital one, and vice-versa?
- How to represent a geometric entity on a digital grid, while preserving its properties?
- Which are the continuous representations of a discrete one?
- Which are the exact intersections of a continuous representation and the digital grid?

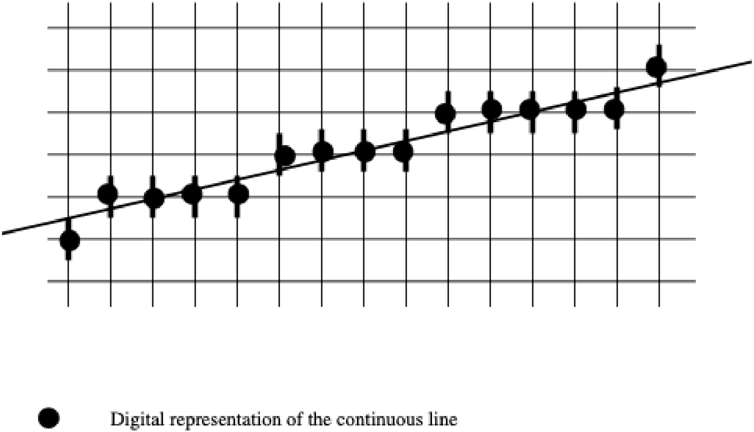
Example of straight lines or segments

Digitization of a continuous line





Digital representation of the continuous line



Cf Bresenham algorithm

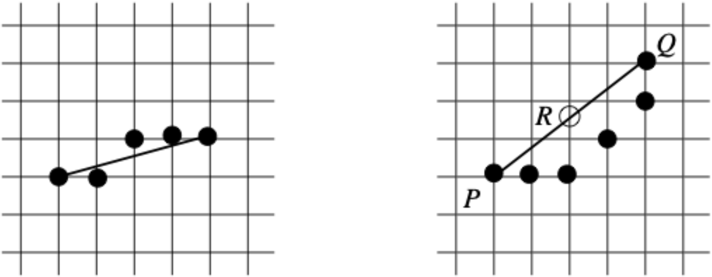
Characterization of a digital straight line segment

Cord property

$\mathcal{S}$ satisfies the cord property iff:

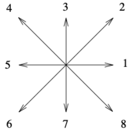
$$\forall (P,Q) \in \mathcal{S}, \forall R \in [P,Q],\ \exists T \in \mathcal{S}, d_\infty(T,R) < 1$$

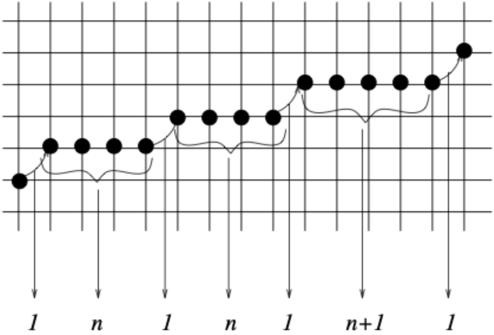
with $d_\infty((x,y),(x',y')) = \max(|x-x'|,|y-y'|)$



Syntactic characterization

- only two “neighbor” directions
- for one direction: sections of length 1
- for the other direction: sections of length n or $n+1$





Analytical digital straight lines

$$y = ax + b$$

Intersections with the grid?

- Condition for non-empty intersection:

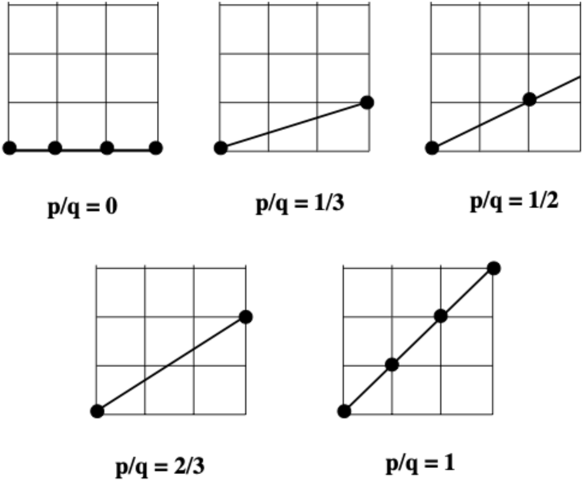
$$a = \frac{p}{q}$$

  p and q integers, co-prime, and:

$$p \leq q \leq N$$

- Farey sequence:
  - image of size $N \times N$ and slope less than 1
  - $\Rightarrow$ possible slopes = Farey sequence of order N: $F(N)$ (cardinality approximately $3N^2/\pi^2$)
  - recursive construction ($\frac{m+m'}{n+n'}$ between $\frac{m}{n}$ and $\frac{m'}{n'}$)

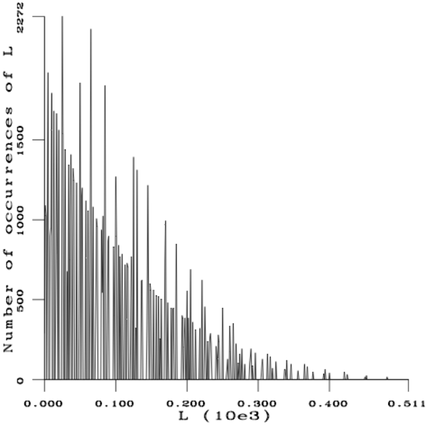
Example for $N = 4$ $(a \leq 1)$: $F(N) = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$



Length of a digital straight segment

$$a^2 + b^2 = L$$

with a and b integer

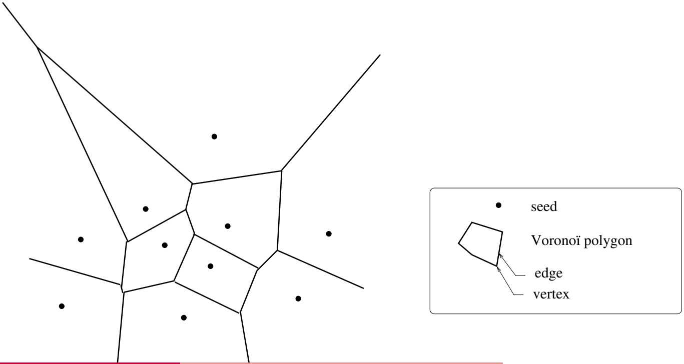


Voronoï diagram

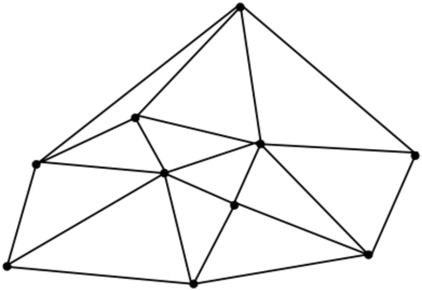
- Useful representation for shapes, image structures...
- Seeds$\{P_1, P_2, ..., P_n\}$
- Voronoï cells:

$$V(P_i) = \{P \in \mathbb{R}^2 \mid \forall j, 1 \leq j \leq n,\ d(P, P_i) \leq d(P, P_j)\}$$

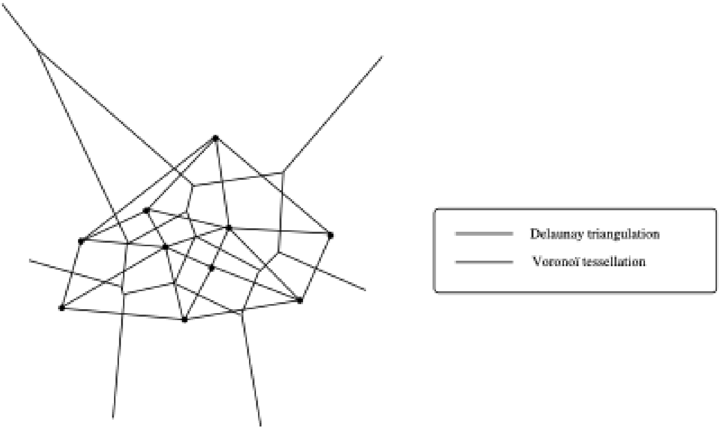
- For the Euclidean distance: $V(P_i) =$ convex polygon



Delaunay triangulation

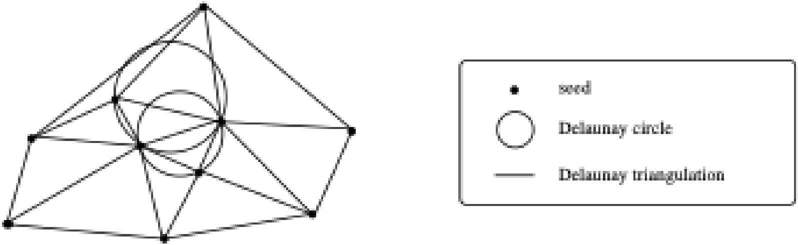


Duality

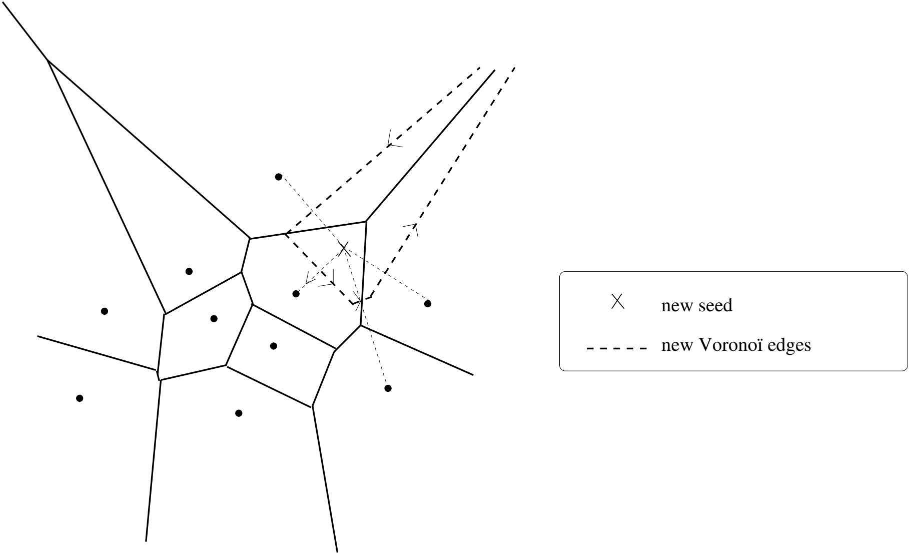


Properties:

- if there are no 4 co-circular points, every Voronoï vertex is equidistant of exactly 3 seeds
- any Voronoï vertex is the center of a circle (called Delaunay circle) passing through 3 seeds and containing no other seed
- $V(P_i)$ is non-bounded iff P_i belongs to the convex hull of the P_js



Incremental construction



A few geometric applications:

- minimal distance between two sets of points
- triangulation such that the circle circumscribed to each triangle is empty
- convex hull of a set of points
- ...

Discrete distances

- $\mathcal{P} = \{\vec{p_1}, ...\vec{p_m}\}$ set of vectors generating a graph
- Associated length d_i
- Conditions:
  - $\vec{p_i} \in \mathcal{P} \Rightarrow -\vec{p_i} \in \mathcal{P}$
  - $\vec{p_i} \in \mathcal{P}, \lambda\vec{p_i} \in \mathcal{P} \Rightarrow \lambda = \pm 1$
  - $||\vec{p_i}|| = ||\vec{p_j}|| \Rightarrow d_i = d_j$

Distance between to vertices / points x and y:

$$d(x, y) = \frac{1}{s} \min\{\sum_{i=1}^{m} n_i d_i \mid n_i \in \mathbb{N}, \sum_{i=1}^{m} n_i \vec{p_i} = \vec{xy}\}$$

s: scale factor

Distance function

Binary image with objects $O \to$ distance map image where the value at x is $d(x, O) = \min_{y \in O} d(x, y)$

- global concept $\Rightarrow$ local computation by propagating local distances
- requirements:
  - good approximation of the Euclidean distance
  - fast algorithms

	1	
1	0	1
	1	

a

1	1	1
1	0	1
1	1	1

b

4	3	4
3	0	3
4	3	4

c

	11		11	
11	7	5	7	11
	5	0	5	
11	7	5	7	11
	11		11	

d

Algorithms

Parallel algorithm

- f^k: image at iteration k
- g: mask
- f^0: points of objects set to 0, points of the background set to $+\infty$

$$f^k(x) = \min\{f^{k-1}(y - x) + g(y),\ y \in \text{support}(g)\}$$

- number of iterations: depends on image size, object size, shape...
- two images in memory
- can be adapted for any grid (2D or 3D) and any mask
- can be parallelized

Sequential algorithm

- two scans of the image, in opposite directions
- masks g_1 and g_2 containing the points already examined according to the scan direction (+ origin)
- f^0: points of objects set to 0, points of the background set to $+\infty$
- for $k = 1, 2$

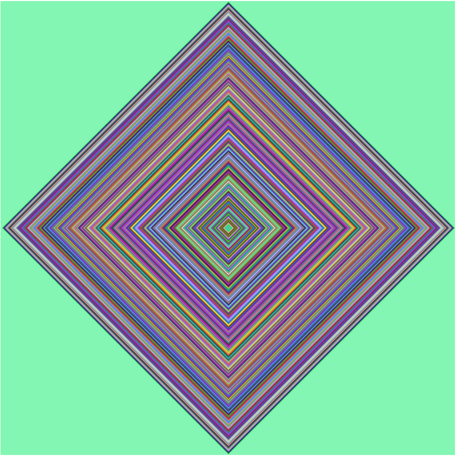
$$f^k(x) = \min\{f^{k-1}(x), f^k(y-x) + g_k(y),\ y \in \text{support}(g_k)\}$$

- fast algorithm
- only one image in memory
- can be adapted for any grid (2D or 3D) and any mask
- recursive

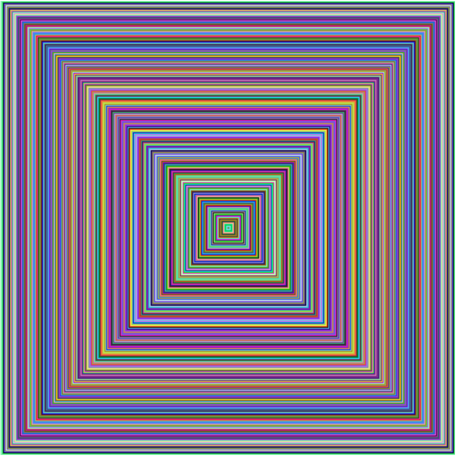
Algorithms based on object contours

- Using chains:
  - contour chaining
  - point displacement and rewriting rules
  - adjustments
- Using queues
  - FIFO initialized with contour points
  - for each point of the queue: computation of the neighbors, distance value increment, and neighbors added in the queue
  - applies in 3D as well

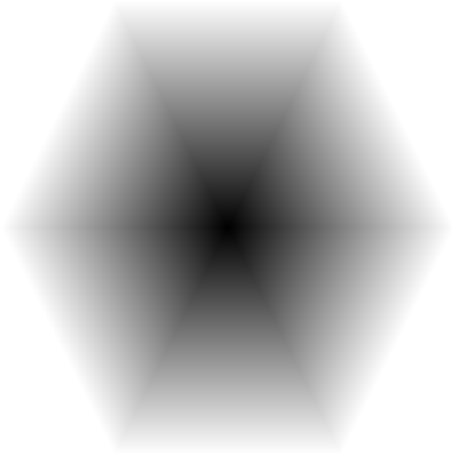
Distance map (4-connectivity mask)

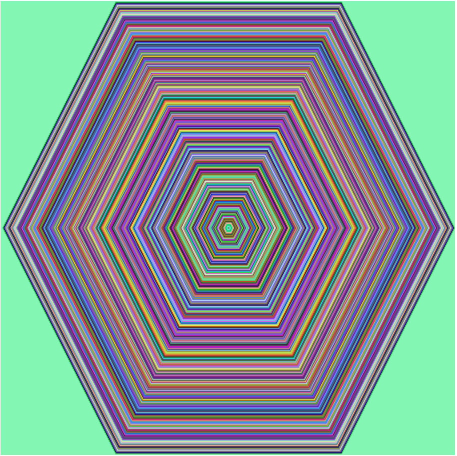


Distance map (8-connectivity mask)

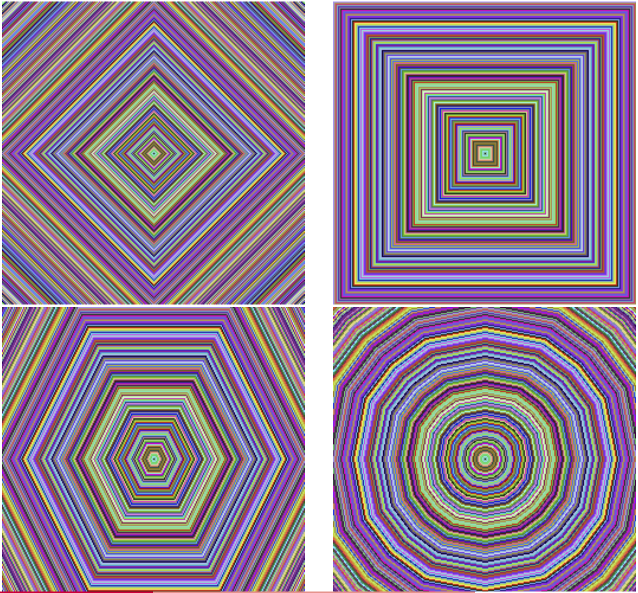


Distance map (6-connectivity mask)

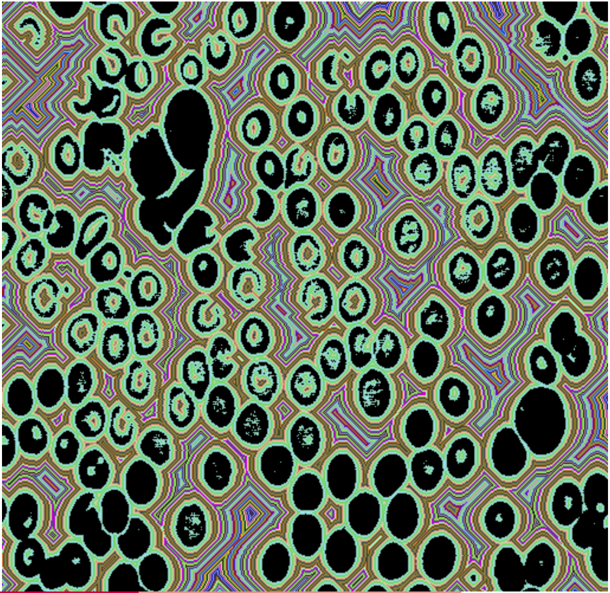


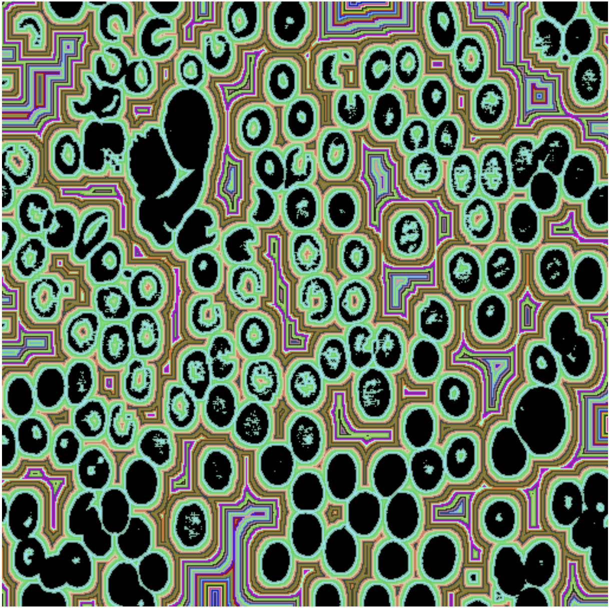


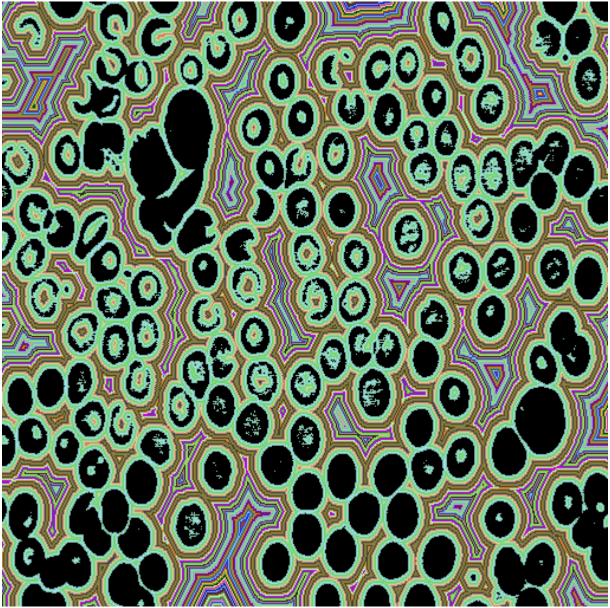
Comparison 4c / 8c / 6c / 5-7-11

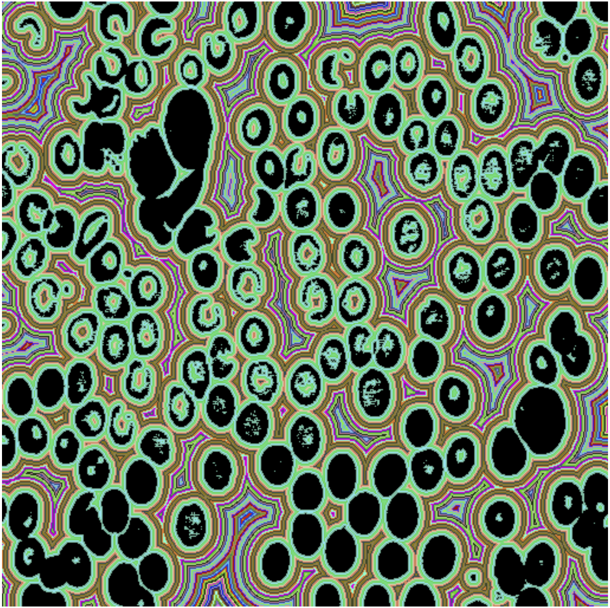


Example on a binarized biological image

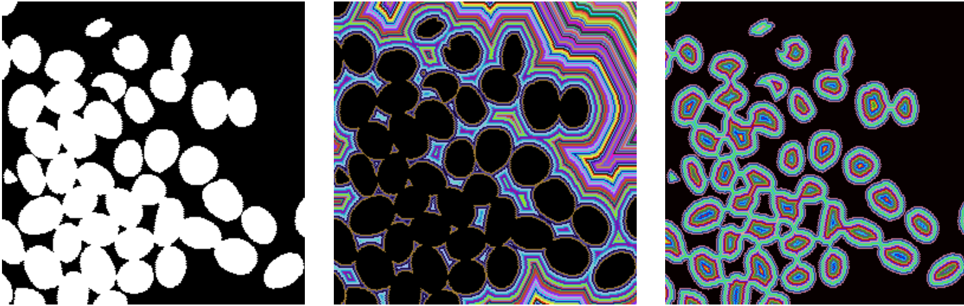




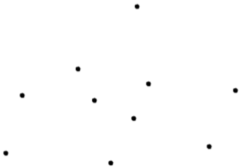


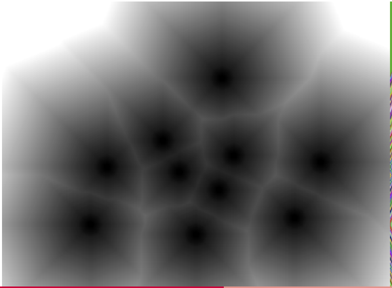


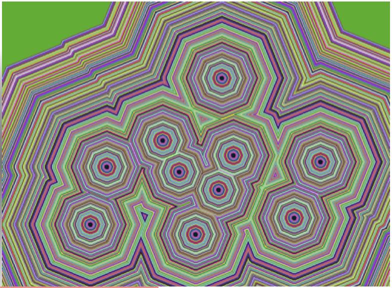
Example on a coffee bean image

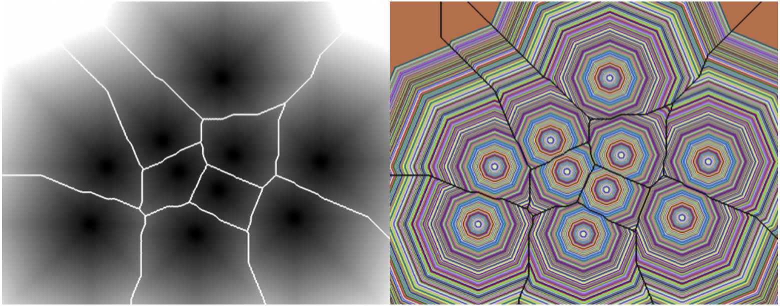


Voronoï diagram from a discrete distance









Applications

- Distance computation (e.g. model-based object recognition, scene understanding)
- Registration
- Mathematical morphology operations on binary images

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