Reasoning: Revision, Merging, Abduction

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A few topics in knowledge representation and reasoning

Logical representation of knowledge and information, and reasoning

- Revision: how to update knowledge or belief based on a new piece of information.
- Merging: how to merge opinions, beliefs, multi-source information.
- Abduction: how to find the best explanation to an observation, based on a knowledge-base.

In all these examples: notion of minimality (e.g. minimal change).

1. Revision

Example:

- Two satellites, Unit A and Unit B, orbit around Mars; the satellites are programmed to land while transmitting their status to Earth.
- Earth has received a transmission from one of the satellites, communicating that it is still in orbit; however, due to interference, it is not known which satellite sent the signal.
- Subsequently, Earth receives the communication that Unit A has landed.

1. Initial set of beliefs: either one of the two satellites is still in orbit.
2. New piece of information: Unit A has landed, and is therefore not in orbit.
3. What is the rational revised belief?

Belief representation in propositional logic

- Langage L
- Usual connectives
- Belief $=$ set K of sentences of L
- New information: ψ
- Revision of K by ψ: $K \star \psi$
- Katsuno and Mendelzon (1991): representation of K as one formula φ ($K = Cn(\varphi)$)
- Revision of φ by ψ: $\varphi \circ \psi$

AGM postulates (Alchourrón, Gärdenfors, Makinson, 1985)

(R1) $\varphi \circ \psi \vdash \psi$ **(Success)**
(R2) If $\varphi \wedge \psi \nvdash \perp$ then $\varphi \circ \psi \equiv \varphi \wedge \psi$ **(Minimality)**
(R3) If $\psi \nvdash \perp$ then $\varphi \circ \psi \nvdash \perp$ **(Coherence)**
(R4) If $\varphi_1 \equiv \varphi_2$ and $\psi_1 \equiv \psi_2$ then $\varphi_1 \circ \psi_1 \equiv \varphi_2 \circ \psi_2$ **(Syntax independence)**
(R5) $(\varphi \circ \psi) \wedge \theta \vdash \varphi \circ (\psi \wedge \theta)$ **(Superexpansion)**
(R6) If $(\varphi \circ \psi) \wedge \theta \nvdash \perp$ then $\varphi \circ (\psi \wedge \theta) \vdash (\varphi \circ \psi) \wedge \theta$ **(Subexpansion)**

Faithful assignment

Mapping which associates to each formula φ a total pre-order $\leq_\varphi$ on the set of models Ω such that:

1. if $\omega \models \varphi$ and $\omega' \models \varphi$ then $\omega \sim_\varphi \omega'$
2. if $\omega \models \varphi$ and $\omega' \models \neg\varphi$ then $\omega <_\varphi \omega'$
3. if $\models \varphi_1 \leftrightarrow \varphi_2$ then $\leq_{\varphi_1} = \leq_{\varphi_2}$

Intuition: qualitative way to express the distance of a world ω to φ, i.e. $\omega \leq_\varphi \omega'$ means that ω is closer to φ than ω'.

Representation theorem (Katusno and Mendelzon)
An operator $\circ$ is a revision operator $\circ$, i.e. that satisfies **R1-R6**, iff there exists a faithful assignment that maps each formula φ to a total pre-order $\leq_\varphi$ such that for every propositional formula ψ we have[1]

$$Mod(\varphi \circ \psi) = \min(Mod(\psi), \leq_\varphi)$$

[1] $\min(A, \leq) = \{\omega \in A \mid \forall \omega' \in A,\ \omega \leq \omega'\}$.

Example: Dalal's revision operator

- d_H: Hamming distance between worlds (counting the number of variables instantiate differently)
- Distance from a world to a formula:

$$d(\omega, \varphi) = \min\{d(\omega, \omega') \mid \omega' \models \varphi\}$$

- Faithful assignment defined from d

$$\omega \leq_\varphi \omega' \text{ iff } d(\omega, \varphi) \leq d(\omega', \varphi)$$

- Associated $\circ =$ Dalal's revision operator.

Example (inspired by Tversky and Kahneman):

- John knew Linda when both of them were PhD students in Philosophy in a very prestigious university. He remembers Linda's activism in feminism, her brilliant record and her dynamism. Both obtained their PhD degree at the same time.
- Since then, five years after, John has no news from Linda. However, he thinks that Linda is for sure an activist in feminism (a), that she occupies an excellent position in a Philosophy Department of one prestigious university (p) and she maintains her dynamism (d).
- Initial belief: $\varphi = a \wedge p \wedge d$.
- John meets Peter, a common classmate, who says him that, surprisingly, Linda is now a bank teller (new information $\psi = \neg p$).
- With this new piece of information John revises his beliefs and he thinks now that Linda is a bank teller who keeps her feminist activism and keeps her dynamism.
- $\varphi \circ \psi = a \wedge d \wedge \neg p$.

2. Merging

Example:

- At a meeting of a block of flat co-owners, the chairman proposes for the coming year the construction of a swimming-pool, of a tennis-court and of a private-car-park.
- The chairman outlines that building two items or more will have an important impact on the rent.
- There are four co-owners. Two of the co-owners want to build the three items and do not care about the rent increase.
- The third one thinks that building any item will cause at some time an increase of the rent and wants to pay the lowest rent so he is opposed to any construction.
- The last one thinks that the block really needs a tennis-court and a private-car-park but does not want a high rent increase.

What would be the best solution?

Objective

Finding a piece of information (or decision) which is

- coherent,
- relevant,
- close enough to each source of information,
- satisfying integrity constraints.

Formalization in PL (Konieczny and Pino Pérez)

- Profile: $E = \{\varphi_1, \ldots, \varphi_n\}$
- Integrity constraints μ
- Merging $\Delta_\mu(E)$
- Rationality postulates:
  (IC0) $\Delta_\mu(E) \vdash \mu$
  (IC1) If μ is consistent, then $\Delta_\mu(E)$ is consistent
  (IC2) If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) \equiv \bigwedge E \wedge \mu$
  (IC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$
  (IC4) If $\varphi_1 \vdash \mu$ and $\varphi_2 \vdash \mu$, then $\Delta_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_1$ is consistent if and only if $\Delta_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_2$ is consistent
  (IC5) $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \vdash \Delta_\mu(E_1 \sqcup E_2)$
  (IC6) If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \vdash \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$
  (IC7) $\Delta_{\mu_1}(E) \wedge \mu_2 \vdash \Delta_{\mu_1 \wedge \mu_2}(E)$
  (IC8) If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \vdash \Delta_{\mu_1}(E)$

$\bigwedge E = \varphi_1 \wedge \ldots \wedge \varphi_n$, $\sqcup$ = multiset union

Syncretic assignment

Function mapping each profile E to a total pre-order $\leq_E$ over interpretations such that for any profiles E, E_1, E_2 and for any belief bases φ, φ':

(1) If $\omega \models E$ and $\omega' \models E$, then $\omega \simeq_E \omega'$
(2) If $\omega \models E$ and $\omega' \not\models E$, then $\omega <_E \omega'$
(3) If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$
(4) $\forall \omega \models \varphi$, $\exists \omega' \models \varphi'$, such that $\omega' \leq_{\varphi \sqcup \varphi'} \omega$
(5) If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega \leq_{E_1 \sqcup E_2} \omega'$
(6) If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$

Representation theorem

An operator Δ is an IC merging operator if and only if there exists a syncretic assignment that maps each profile E to a total pre-order $\leq_E$ such that

$$Mod(\Delta_\mu(E)) = \min(Mod(\mu), \leq_E)$$

Construction based on a distance and on an aggregation function

- Pseudo-distance d between interpretation, i.e.
  - $\forall \omega, \omega', d(\omega, \omega') = d(\omega', \omega)$
  - $d(\omega, \omega') = 0$ iff $\omega = \omega'$
- Aggregation function f such that for any $x_1, \ldots, x_n, x, y \in \mathbb{R}^+$:
  - monotony: if $x \leq y$, then $f(x_1, \ldots, x, \ldots, x_n) \leq f(x_1, \ldots, y, \ldots, x_n)$
  - minimality: $f(x_1, \ldots, x_n) = 0$ iff $x_1 = \ldots = x_n = 0$
  - identity: $f(x) = x$
- $d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$
- $d(\omega, E) = f(d(\omega, \varphi_1) \ldots, d(\omega, \varphi_n))$
- $\omega \leq_E \omega'$ iff $d(\omega, E) \leq d(\omega', E)$ is a syncretic assignment if f has additional properties:
  - symmetry: for any permutation σ, $f(x_1 ... x_n) = f(\sigma(x_1) ... \sigma(x_n))$
  - composition: if $f(x_1 ... x_n) \leq f(y_1 ... y_n)$, then $f(x_1 ... x_n, z) \leq f(y_1 ... y_n, z)$
  - decomposition: if $f(x_1 ... x_n, z) \leq f(y_1 ... y_n, z)$ then $f(x_1 ... x_n) \leq f(y_1 ... y_n)$
- $Mod(\Delta_\mu(E)) = \min(Mod(\mu), \leq_E)$

Examples

- $d = d_H$ (Hamming distance)
- $\Delta^{d,\Sigma}$ from $f = \Sigma$
- $\Delta^{d,Gmax}$ from $f = leximin$

Example (R. Pino Pérez)

- S *swimming-pool*, T *tennis-court*, P *private-car-park and* I *rent increasing.*
- $\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I.$
- $K_1 \equiv K_2 \equiv S \wedge T \wedge P$
- $K_3 \equiv \neg S \wedge \neg T \wedge \neg P \wedge \neg I.$
- $K_4 = T \wedge P \wedge \neg I.$
- $E = \{K_1, K_2, K_3, K_4\}$ *(profile).*
- $\Delta_\mu(E) =$??

  (the result of merging E under the integrity constraint μ*)*

$[\![K_1]\!] = [\![K_2]\!] = \{1110, 1111\}, [\![K_3]\!] = \{0000\}, [\![K_4]\!] = \{0110, 1110\}$

	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	Σ	GMAX
$(0, 0, 0, 0)$	3	3	0	2	8	(3,3,2,0)
$(0, 0, 0, 1)$	3	3	1	3	10	(3,3,3,1)
$(0, 0, 1, 0)$	2	2	1	1	6	**(2,2,1,1)**
$(0, 0, 1, 1)$	2	2	2	2	8	(2,2,2,2)
$(0, 1, 0, 0)$	2	2	1	1	6	**(2,2,1,1)**
$(0, 1, 0, 1)$	2	2	2	2	8	(2,2,2,2)
$(0, 1, 1, 0)$	1	1	2	0	4	(2,1,1,0)
$(0, 1, 1, 1)$	1	1	3	1	6	(3,1,1,1)
$(1, 0, 0, 0)$	2	2	1	2	7	(2,2,2,1)
$(1, 0, 0, 1)$	2	2	2	3	9	(3,2,2,2)
$(1, 0, 1, 0)$	1	1	2	1	5	(2,1,1,1)
$(1, 0, 1, 1)$	1	1	3	2	7	(3,2,1,1)
$(1, 1, 0, 0)$	1	1	2	1	5	(2,1,1,1)
$(1, 1, 0, 1)$	1	1	3	2	7	(3,2,1,1)
$(1, 1, 1, 0)$	0	0	3	0	3	(3,0,0,0)
$(1, 1, 1, 1)$	0	0	4	1	**5**	(4,1,0,0)

$$[\![\Delta_\mu^\Sigma(E)]\!] = \{(1, 1, 1, 1)\}$$

$$[\![\Delta_\mu^{\mathrm{GMAX}}(E)]\!] = \{(0, 0, 1, 0), (0, 1, 0, 0)\}$$

Links between revision and merging

- If Δ is an IC merging operator (satisfying **IC0–IC8**), then the operator $\circ$, defined as

$$\varphi \circ \mu = \Delta_\mu(\varphi)$$

  is an AGM revision operator (satisfying **R1–R6**).

Links between belief merging (BM) social choice theory (SCT)

Notions in BM	Belief Merging	SCT	Notions in SCT
Agents	i	i	Voters
Belief bases	φ_i	$\succeq_i$	Individual preferences
Profiles (of belief bases)	$\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$	$p = (\succeq_1, \succeq_2, \ldots, \succeq_n)$	Profiles (of preferences)
Integrity constraints	μ	A	Agendas
Assignments	$\Phi \mapsto \succeq_\Phi$	$p \mapsto \succeq_p$	social welfare function
Result of the merging	$[\![\Delta_\mu(\Phi)]\!] = \min([\![\mu]\!], \preceq_\Phi)$	$F(p, V) = \max(V, \succeq_p)$	Alternatives chosen

3. Abduction

Introduced by Charles Sanders Peirce (1839-1914).

Example

- Knowledge:
  - If it rained last night, then grass is wet.
  - If sprinkler is on, then grass is wet.
- Observation:
  - Grass is wet.
- What explains this observation?

Formalization in logic $\alpha \rhd \gamma$

Explanation γ of observation α according to a knowledge base (KB) or a theory Σ (assuming $\Sigma \cup \{\alpha\}$ is consistent) should satisfy:

- $\Sigma \cup \{\gamma\}$ consistent
- $\Sigma \cup \{\gamma\} \vdash \alpha$ or $\Sigma \cup \{\gamma\} \models \alpha$
- γ is "as simple as possible"
- in some approaches: $\gamma \not\models \alpha$ (debatable), $\Sigma \not\models \alpha$

Note that adding formulas to Σ reduces the set of models.

$$\mathit{Expla}_{\Sigma}(\alpha) = \{\gamma \mid \Sigma \cup \{\gamma\} \not\vdash \bot, \Sigma \cup \{\gamma\} \models \alpha\}$$

Example: formalization in PL

- $\Sigma = \{r \rightarrow w, s \rightarrow w\}$
- $\alpha = w$
- $Expla_{\Sigma}(\alpha) = \{r, s, r \wedge s, r \vee s, w, r \wedge w ...\}$
- Minimality (simplicity): r may be preferred to $r \wedge s$

Minimality criteria

- Subset minimality ($\subseteq$), or irredundancy: solution S such that no proper subset $S' \subset S$ is a solution.
- Minimum cardinality ($\leq$): solution S is preferred to solution S' is $|S| < |S'|$.
- Priority levels.
- Penalities: priorities combined with $\leq$-minimality.

Rationality postulates (Pino Perez and Uzcategui)

LLE$_\Sigma$:	If $\vdash_\Sigma \alpha \leftrightarrow \alpha'$ and $\alpha \rhd \gamma$ then $\alpha' \rhd \gamma$.
RLE$_\Sigma$:	If $\vdash_\Sigma \gamma \leftrightarrow \gamma'$ and $\alpha \rhd \gamma$ then $\alpha \rhd \gamma'$.
E-CM:	If $\alpha \rhd \gamma$ and $\gamma \vdash_\Sigma \beta$ then $(\alpha \wedge \beta) \rhd \gamma$.
E-C-Cut:	If $(\alpha \wedge \beta) \rhd \gamma$ and $\forall \delta\ [\alpha \rhd \delta \ \Rightarrow \ \delta \vdash_\Sigma \beta\ ]$ then $\alpha \rhd \gamma$.
RS:	If $\alpha \rhd \gamma$, $\gamma' \vdash_\Sigma \gamma$ and $\gamma' \not\vdash_\Sigma \bot$ then $\alpha \rhd \gamma'$.
ROR:	If $\alpha \rhd \gamma$ and $\alpha \rhd \delta$ then $\alpha \rhd (\gamma \vee \delta)$.
LOR:	If $\alpha \rhd \gamma$ and $\beta \rhd \gamma$ then $(\alpha \vee \beta) \rhd \gamma$.
E-DR:	If $\alpha \rhd \gamma$ and $\beta \rhd \delta$ then $(\alpha \vee \beta) \rhd \gamma$ or $(\alpha \vee \beta) \rhd \delta$.
E-R-Cut:	If $(\alpha \wedge \beta) \rhd \gamma$ and $\exists \delta\ [\alpha \rhd \delta\ \&\ \delta \vdash_\Sigma \beta]$ then $\alpha \rhd \gamma$.
E-Reflexivity:	If $\alpha \rhd \gamma$ then $\gamma \rhd \gamma$.
E-Con$_\Sigma$:	$\not\vdash_\Sigma \neg\alpha$ iff there is γ such that $\alpha \rhd \gamma$.

Tableaux method

= an example of computational procedure

- Tableau = binary tree
- built from an initial set of formulas
- using construction rules

Main idea:

$$T \models \varphi?$$

Tableau from $T \cup \{\neg\varphi\}$

- if closed (every branch contains a variable and its negation): the initial set is unsatisfiable, and $T \models \varphi$ holds;
- if open branches, φ is not a valid consequence of T.

Application for abduction: build tableau from $\Sigma \cup \{\neg\alpha\}$ and find how to close open branches.

+ Minimality constraint

Construction (or expansion) rules

For l_1, l_2 literals:

- $(l_1 \wedge l_2) \Longrightarrow (l_1, l_2)$
- $(l_1 \vee l_2) \Longrightarrow (l_1 \mid l_2)$
- $(l_1 \rightarrow l_2) \Longrightarrow (\neg l_1 \mid l_2)$

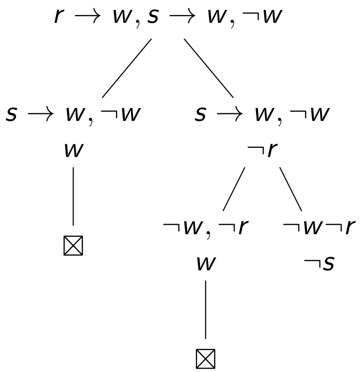
where $|$ indicates two separated branches

- $\neg\neg l_1 \Longrightarrow l_1$
- $\neg(l_1 \wedge l_2) \Longrightarrow \neg l_1 \vee \neg l_2$
- $\neg(l_1 \vee l_2) \Longrightarrow \neg l_1 \wedge \neg l_2$

Branch = decomposition sequence until a node with only atomic propositions and their negations is reached.

Example: abduction with tableaux

- $\Sigma = \{r \to w, s \to w\}$
- $\alpha = w$
- Tableau from $\Sigma \cup \{\neg\alpha\}$:

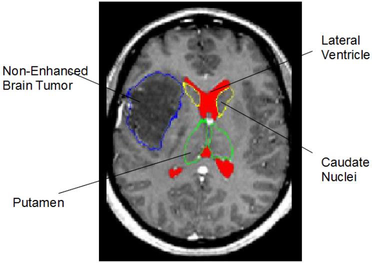


- Formulas that close the open branch: $\{r, s, r \wedge s, r \vee s, w, r \wedge w \ldots\}$

Other methods:

- Truth tables
- Prime implicates:
  - Clause C = prime implicate of a KB Σ if $\Sigma \models C$ and $\forall C', \Sigma \models C'$ and $C' \models C$ implies $C' \equiv C$.
  - Find C that contains φ to be explained.
  - Explanations $= \neg(C \setminus \varphi)$.
- Sequent calculus

A more complex example:



Pathological brain with a tumor

$$\mathcal{K} \models (\gamma \rightarrow \mathcal{O})$$

Compute the "best" explanation to the observations taking into account the expert knowledge (e.g. formalized in description logic).
$\Rightarrow$ Spatial reasoning

Extensions for revision, merging, abduction

- Other postulates
- Concrete constructions
- Other logics:
  - Description logics
  - Fuzzy logics
  - Modal logics
  - Satisfaction systems and institutions